Lecture 19: Basic graph theory

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1 Adjacency matrix

Lemma 1 (Matrix shift). *Suppose* $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ *are the eigenvalues of* M*. Then the eigenvalues of* $\lambda I + M$ *are* $\{\lambda + \lambda_1, \lambda + \lambda_2, \cdots, \lambda + \lambda_n\}$*, where* I *is the identity matrix. The converse is also true.*

Recall that a symmetric real matrix A has only real eigenvalues. (Hint: $(u^* A u)^* = u^* A^* u = u^* A u$, for all vectors u.)

Theorem 1. *The maximum eigenvalue of the adjacency matrix of a regular graph* G *is* d*, the degree of the graph.*

Proof. By using Lemma 1, it is sufficient to prove that all eigenvalues of $dI - A_G$ are greater than zero.

For the sake of contradiction, assume that $M = dI - A_G$ has a negative eigenvalue μ with eigenvector u. Then $u^T M u = \mu u^T u < 0$. We will prove that for every vector v, $v^T M v \geq 0$, hence get a contradiction.

Let v_i be the i-th entry of v and M_{ij} be the (i,j)-th entry of M. Then,

$$\begin{aligned} v^T M v &= \sum_{i,j} M_{ij} v_i v_j \\ &= \sum_i d \cdot (v_i)^2 - \sum_{(i,j)\in E_G} v_i v_j \\ &= \sum_{(i,j)\in E_G} (v_i - v_j)^2 \geq 0 \quad \text{[Count } v_i^2\text{]}. \end{aligned} \tag{1}$$

Note 1. The term $v^T M v$ is known as the *quadratic form* of M.

To prove that d is an eigenvalue, notice that every row of the matrix A_G sums up to d.

Exercise 1. What is the eigenvector corresponding to the eigenvalue d ?

□

A graph is called *bipartite* if the vertex set can be divided into two parts A and B, s.t., there are no edges inside A and no edges inside B. A regular bipartite graph can be characterized by its eigenvalues.

Theorem 2. *The minimum eigenvalue of the adjacency matrix of a* d*-regular, connected graph is greater than or equal to* $-d$*.*

It is bipartite if and only if the minimum eigenvalue is $-d$*.*

Proof. By using Lemma 1, it is sufficient to prove that all eigenvalues of $M := dI + A_G$ are greater than or equal to zero. We will prove that: There will be an eigenvalue 0 iff the graph is bipartite.

* Edited from Rajat Mittal's notes.

Again we will notice the quadratic form $v^T(dI + A_G)v$,

$$\begin{aligned} v^T M v &= \sum_{i,j} M_{ij} v_i v_j \\ &= \sum_i d(v_i)^2 + \sum_{(i,j)\in E_G} v_i v_j \\ &= \sum_{(i,j)\in E_G} (v_i + v_j)^2 \geq 0 \quad [\text{Count } v_i^2]. \end{aligned} \tag{2}$$

Suppose equality holds in the above equation. Notice that v_i corresponds to the i-th vertex. If for some i, $v_i = 0$ then, by repeatedly using the identities $v_i + v_j = 0$ for $(i,j) \in E_G$, we deduce that v is the zero vector. Thus, for all i, $v_i \neq 0$.

Let us say S_1 is the set of vertices for which v_i is positive and S_2 is the set of vertices for which v_i is negative. Then there are no edges inside S_1 and inside S_2.

This implies that the graph is bipartite, with $V(G)$ partitioned into S_1 and S_2.

Conversely, if the graph is bipartite, then assign 1 to one part and -1 to the other part of the vertices. This will show that the least eigenvalue of $dI + A_G$ is zero (why?). Thus, the least eigenvalue of A_G is $-d$.

□

Exercise 2. Show that the bipartite graph cannot have an odd cycle.

A *complete graph* is a simple graph where every possible edge is present. A complete graph on n vertices is called K_n.

Exercise 3. How many edges are there in a complete graph?

The adjacency matrix of the complete graph K_n is $J - I$, where J is all 1's matrix and I is the identity matrix, both of dimension $n \times n$.

Exercise 4. What are the eigenvalues of the adjacency matrix of the complete graph.

$(n-1)$ once and the rest are -1's.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.
2. N. L. Biggs. Discrete Mathematics. *Oxford University Press*, 2003.