Lecture 18: Basic graph theory

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1 Hamiltonian cycle

Similar to Euler circuits, a path $x_0, x_1 \cdots, x_k$ is called a *Hamiltonian path* if it goes through all the vertices of the graph. Remember that in a path we are not allowed to repeat the vertices.

It might seem like an easy task to give a necessary and sufficient condition for the existence of Hamiltonian path like Eulerian circuit, but it turns out to be a really hard problem.

You will study later that finding a Hamiltonian path in a graph is an NP-complete problem (the list of problems which are assumed to be hard for computers to solve efficiently).

There are many sufficient conditions known for existence and non-existence of Hamiltonian path.

Exercise 1. Show that a complete graph has a Hamiltonian path.

Exercise 2. If the first and the last vertex of a Hamiltonian path is same then it is called a *Hamiltonian cycle*. Show that if a graph has a vertex of degree one then it cannot have a Hamiltonian cycle.

2 Adjacency matrix

Adjacency matrix is a representation of a graph in matrix format. Given a graph G, its adjacency matrix A_G is a $|V| \times |V|$ matrix. Its rows and columns are indexed by the vertices of the graph. The (i, j)-th entry of the matrix is one if and only if vertex i is adjacent to vertex j, otherwise it is zero.

The adjacency matrix for the graph in Fig. 1 is,

$$A_G = \begin{pmatrix} 0\,0\,0\,1\,0\,1 \\ 0\,0\,1\,0\,0\,1 \\ 0\,1\,0\,0\,1\,0 \\ 1\,0\,0\,0\,1\,0 \\ 0\,0\,1\,1\,0\,0 \\ 1\,1\,0\,0\,0\,0 \end{pmatrix}$$

Note 1. The adjacency matrix for an undirected graph is always symmetric.

Another matrix representation of a graph is called the *incidence matrix*. Here the rows are indexed by vertices and columns are indexed by edges. An entry (i, e) is one if and only if vertex i is part of edge e, otherwise it is zero. The incidence matrix for the graph in Fig. 1 is,

	e_1	e_2	e_3	e_4	e_5	e_6
v_1	1	0	0	0	0	1
v_2	0	0	1	1	0	0
v_3	0	0	0	0	1	1
v_4	1	1	0	0	0	0
v_5	0	1	0	0	1	0
v_6	0	0	1	0	0	1

⋆ Edited from Rajat Mittal's notes.

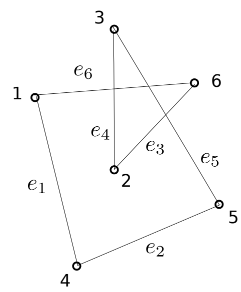

Fig. 1. A simple graph

Note 2. Here v_i represents the i-th vertex.

The eigenvalues and eigenvectors of the adjacency matrix provide us with a lot of information about the graphs. We will see a few examples below. First, a lemma which will help us in proving results about eigenvalues.

Lemma 1 (Matrix shift). *Suppose $\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ are the eigenvalues of M. Then the eigenvalues of $\lambda I + M$ are $\{\lambda + \lambda_1, \lambda + \lambda_2, \cdots, \lambda + \lambda_n\}$, where I is the identity matrix. The converse is also true.*

Proof. Exercise in linear algebra. □

Exercise 3. The eigenvalues of a symmetric real matrix are real.

Hence, adjacency matrices have real eigenvalues.

A graph is called *regular* if every vertex has the same degree d. In this case, d is called the degree of the graph.

Theorem 1. *The maximum eigenvalue of the adjacency matrix of a regular graph G is d, the degree of the graph.*

Proof. By using Lemma 1, it is sufficient to prove that all eigenvalues of $dI - A_G$ are greater than zero.

For the sake of contradiction, assume that $M = dI - A_G$ has a negative eigenvalue μ with eigenvector u. Then $u^T M u = \mu u^T u < 0$. We will prove that for every vector v, $v^T M v \geq 0$, hence get a contradiction.

Let v_i be the i-th entry of v and M_{ij} be the (i, j)-th entry of M. Then,

$$\begin{aligned} v^T M v &= \sum_{i,j} M_{ij} v_i v_j \\ &= \sum_i d \cdot (v_i)^2 - \sum_{(i,j) \in E_G} v_i v_j \\ &= \sum_{(i,j) \in E_G} (v_i - v_j)^2 \geq 0 \quad [\text{Count } v_i^2]. \end{aligned} \tag{1}$$

Note 3. The term $v^T M v$ is known as the *quadratic form* of M.

To prove that d is an eigenvalue, notice that every row of the matrix A_G sums up to d.

Exercise 4. What is the eigenvector corresponding to the eigenvalue d ?

□

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.