Lecture 17: Basic graph theory

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1 Trees

A graph is called a *tree* if,

- It is connected, and
- There are no cycles in the graph.

Tree is a special graph where every pair of vertices have a unique path between them. Let us prove this formally.

Theorem 1. *A graph G is a tree iff there is a* unique *path between every pair of vertices in G.*

Proof. If there is a unique path between every pair of vertices then the graph is connected. Suppose such a graph has a cycle $x_0, x_1, \cdots, x_k$. Then there are two distinct paths between x_0 and x_1 (why?). So a graph with unique path between every pair is a tree.

For the converse, in a tree suppose there are two paths between vertices u and v. Say $P = \{u = x_0, x_1, \cdots, x_k = v\}$ and $P' = \{u = y_0, y_1, \cdots, y_k = v\}$. Let i be the first index, s.t., $x_{i+1} \neq y_{i+1}$. Similarly j be the last index, s.t., $x_{j-1} \neq y_{j-1}$.

Exercise 1. Show that i and j exist and $j - i \geq 2$.

Then consider the walk $x_i, x_{i+1}, \cdots, x_j = y_j, y_{j-1}, \cdots, y_{i+1}, y_i$. This is a walk with first and last vertex being the same and no two consecutive edges are same. So there is a cycle in this tree. Contradiction. ☐

Exercise 2. Show that if you remove any edge of a tree, you will get two disjoint connected components of the graph.

Exercise 3. A tree always has a vertex of degree one.

Assume that there is no degree one vertex. Start walking from a vertex s, in a tree T, in such a way that on reaching any vertex u you go out taking an edge different from the one used to reach u. Since there is no cycle in T, this walk will never end!

Then using induction on number of vertices, we can show that,

Theorem 2. *A tree on n vertices has $n - 1$ edges.*

Pick a degree one vertex v in the tree T. Consider the subtree $T \setminus \{v\}$.

If a subgraph T of a graph G is a tree, on $V(G)$, then T is called a *spanning tree* of the graph G. If the graph G is connected, we can always construct a spanning tree. Define S_0 to be the initial set containing a particular vertex v. At every stage, construct S_{i+1} by including an edge which connects S_i to some vertex not in S_i.

Exercise 4. Why does this not create a cycle?

The process ends when S_i has all the vertices of the graph. We can proceed with each stage (find a vertex to add to S_i) because the graph is connected.

Spanning trees are important in many applications. They are the smallest structure which preserve the connectivity. If we assign weights/cost to every edge of the graph then we are mostly interested in *minimum weight* spanning tree. You will study algorithms to find minimum weight spanning tree in a graph in future courses.

* Edited from Rajat Mittal's notes.

2 Eulerian circuit

A *simple circuit* is a walk where first and last point are the same and no edges are repeated. An *Euler's circuit* is a simple circuit which uses all possible edges of the graph.

Note 1. The vertices can be repeated

The definition of Euler circuits arose from the problem of "Königsberg bridges".

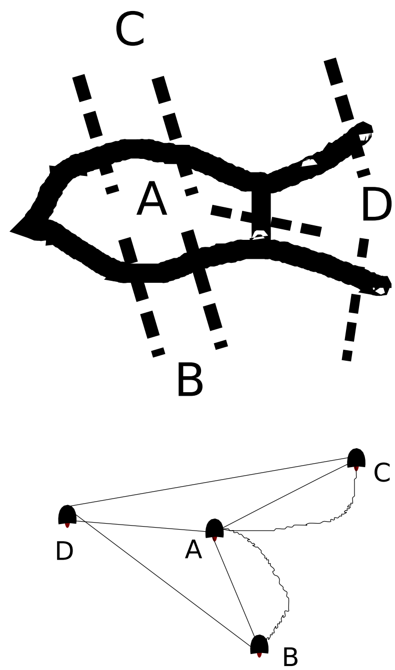

Fig. 1. Königsberg bridges and their graph representation. Dashed lines represent the bridges.

There are four regions connected with seven bridges. Can you go through all the bridges without revisiting any bridge? Converting it into graph as shown in Fig. 1, the question is equivalent to finding an Eulerian circuit in the graph.

The following theorem gives the answer.

Theorem 3. *A connected graph (not necessarily simple) has an Eulerian circuit iff all vertices have even degree.*

Proof. If the graph has an Eulerian circuit then every vertex has even degree (show it as an exercise). Let us prove that if every vertex has an even degree then there is an Eulerian circuit.

We will construct the Eulerian circuit recursively. Start with an arbitrary vertex, say w, find an edge and move in that direction. Since the degree is even, whenever we arrive at a particular vertex, we have a different edge to leave that vertex too.

Since the number of edges are finite, we will arrive back at the starting vertex (call this cycle C). If all edges are covered then we have found the Eulerian circuit. If not, call the subgraph with edges of C removed as H. All the connected components of H will be connected to cycle C (since the graph is connected). Suppose the connected components are $H_1, H_2, \cdots, H_k$ and connection points are $w_1, w_2, \cdots, w_k$ when traversing the cycle C from w in a particular direction (say anticlockwise).

Construct the Eulerian circuit in all connected components of H (Note: each vertex there still has even degree). We can use induction on number of edges, since H_i has less number of edges than the original graph. The base case, total 4 edges in graph, is easy.

Then the Eulerian circuit for G will be, go from w to w_1 using edges of C, take the Eulerian circuit of H_1, go from w_1 to w_2, take the Eulerian circuit for H_2, $\cdots$, reach w_k and take the Eulerian circuit for H_k, come back to w using remaining edges of C.

$\square$

Exercise 5. Write the pseudocode for finding a Eulerian circuit in a graph G.

Exercise 6. Show that it is not possible to take an Eulerian walk on the Königsberg bridges.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.