Lecture 16: Basic graph theory

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1 Paths and cycles

An important question in graph theory is that of connectivity. Given a graph, can we reach from a source vertex s to target vertex t using the edges of the graph?

A *walk* of length k in a graph is a sequence of vertices $x_0, x_1, \cdots, x_k$, where any two consecutive vertices are connected by an edge in the graph. In a walk, it is allowed to take an edge or a vertex multiple times.

If all the vertices in a walk are distinct, except possibly the first and the last vertex, then it is called a *path*. In some texts, paths are called simple paths and walks are called paths.

A path of length greater than two is called a *cycle* if the first and the last vertex are the same.

Example 1. Consider the simple graph $V(G) = [4]$, $E(G) = \{(1,2),(2,3),(3,4),(4,1),(4,2)\}$. The sequence $(1,2,3,4,1)$ is a walk, path and a cycle.

On the other hand, the sequence $(1,2,4,3,2,1)$ is a walk, but not a path or cycle.

The sequence $(1,2,3,4)$ is a walk and a path, but not a cycle.

Lemma 1. *If there is a walk between two vertices in a graph then there is a path between them.*

Proof. Consider the walk of *least* length, say $P = \{x_0, x_1, \cdots, x_k\}$. If all vertices are distinct then it is a path.

Suppose that there is a vertex v which occurs twice in the walk. If we delete the portion of the walk between the two occurrences of the vertex v, we still get a walk P' from x_0 to x_k.

But P was the smallest walk from x_0 to x_k, this is a contradiction. Thus, P is already a path. □

Exercise 1. Show that if there is a walk with the same first and last vertex and no two consecutive edges are the same, then there is a cycle in the graph.

Let the walk be $P = \{x_0, x_1, \cdots, x_k = x_0\}$. By the hypothesis, it can be seen that $k \geq 3$. As before, if there is a vertex $v \neq x_0$ which occurs twice in the walk, then we *delete* the portion of the walk between the two occurrences of v. In the end, P becomes a *path* from x_0 to x_0 of length at least 3. Hence, P is a cycle.

A graph is called *connected* if there is a path between every pair of vertices of the graph.

A graph can always be divided into disjoint parts which are connected within themselves but not connected to each other. These are called the *connected components* of the graph. They are *uniquely* determined.

Exercise 2. Show that if there is path between a particular vertex v and every other vertex of the graph, then the graph is connected.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.

* Edited from Rajat Mittal's notes.