Lecture 15: Basic graph theory

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Combinatorial graphs provide a natural way to model connections between different objects. They are very useful in depicting communication networks, social networks and many other kind of networks. For the purpose of this course, graphs will mean combinatorial (or abstract) graphs as opposed to graphs of functions etc. which you might have learnt previously.

Graphs have become such an important tool, that a complete field, *Graph Theory*, is devoted to learning about the properties of graphs. In the next few lectures we will learn about graphs, associated concepts and various properties of graphs.

1 Definition

A *graph*, $G = (V, E)$, is described by a set of vertices V and edges E which represent the connection between the vertices ($E \subseteq V \times V$). If (u, v) is an element in E then we say that vertex u is *connected* to vertex v. We also say that u is *adjacent* to v. The vertex set of a graph G is represented by $V(G)$ and the edge set by $E(G)$.

The graph is called *simple* if,

- there is no loop, i.e., no vertex is connected to itself,
- there is only one edge between any two vertices,
- if there is no direction assigned to the edges.

If there is a possibility of multiple edges between vertices then it is called a *multigraph*. If the edges are assigned directions, i.e., (u, v) is ordered then it is called a *directed* graph as opposed to an *undirected* graph.

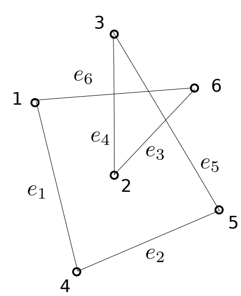

Fig. 1. A simple graph

Graphs are used extensively in modeling social networks. The members of the social group are represented by vertices and their relationship is modeled by the edges between them. If the graph depicts the friendship

* Edited from Rajat Mittal's notes.

between people, like Facebook, it is an undirected graph. If it has a relation which has direction (say a is elder to b in a family), then we have a directed graph.

Another example of a directed graph is how different processes are run on a computer. Suppose a process a can be run only if process b is completed. This is called the *precedence graph* and a directed edge (u, v) shows that u should run before v.

We will study undirected graphs in this course. Note that many of the concepts described below can be studied for the directed version too.

A graph $H = (V, E')$ is called the *subgraph* of a graph $G = (V, E)$ if the edge set E' is a subset of the edges E in G. In other words, some edges of G are not present in H, but every edge of H is present in G.

The *degree* of a vertex v in a graph G is the number of edges from v in G. The degree of a vertex cannot be greater than $|V| - 1$. By double counting, we can easily prove the following theorem.

Theorem 1. *The sum of the degrees of all the vertices in a graph is equal to twice the number of edges.*

Count the number of pairs $\{(u, v) \in V^2 \mid (u, v) \in E(G)\}$. By looking at the degrees we get $\sum_{u \in V(G)} \deg(u)$. It is also, simply, $2 \cdot |E|$.

One of the major problems in theoretical computer science is to figure out if two given graphs are isomorphic. Two graphs, G and H, are called *isomorphic* if there exists a bijection $\pi : V(G) \to V(H)$, s.t., $(u, v) \in E(G)$ if and only if $(\pi(u), \pi(v)) \in E(H)$.

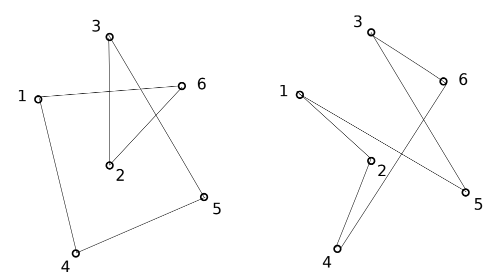

Fig. 2. Are these two graphs isomorphic?

Exercise 1. Construct all possible non-isomorphic connected graphs on four vertices with at most 4 edges.

Exercise 2. Construct two graphs which have same degree set (set of all degrees) but are not isomorphic.

2 Paths and cycles

An important question in graph theory is that of connectivity. Given a graph, can we reach from a source vertex s to target vertex t using the edges of the graph?

A *walk* of length k in a graph is a sequence of vertices $x_0, x_1, \cdots, x_k$, where any two consecutive vertices are connected by an edge in the graph. In a walk, it is allowed to take an edge or a vertex multiple times.

If all the vertices in a walk are distinct, except possibly the first and the last vertex, then it is called a *path*. In some texts, paths are called simple paths and walks are called paths.

A path of length greater than two is called a *cycle* if the first and the last vertex are the same.

Example 1. Consider the simple graph $V(G) = [4]$, $E(G) = \{(1,2),(2,3),(3,4),(4,1),(4,2)\}$. The sequence $(1,2,3,4,1)$ is a walk, path and a cycle.

On the other hand, the sequence $(1,2,4,3,2,1)$ is a walk, but not a path or cycle.

The sequence $(1,2,3,4)$ is a walk and a path, but not a cycle.

References

1. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.