Lecture 9: Counting, Pigeonhole principle

Nitin Saxena *

IIT Kanpur

Exercise 1. The number of partitions of a set with n elements is called the *Bell number*, B_n. Show that it satisfies the recurrence,

$$B_n = \sum_{i=1}^{n} \binom{n-1}{i-1} B_{n-i}\,.$$

This is equal to the number of equivalence relations on a set with n elements.

Also, this is the case of putting n distinct balls into n identical bins.

Define $B_0 := 1$. Let the n elements be $[n]$. In a partition, let $i \in [n]$ be the size of the part that contains n. The number of such partitions is $\binom{n-1}{i-1} \cdot B_{n-i}$. The recurrence obtained is a good case for using exponential generating function:

$$\frac{B_n}{(n-1)!} = \sum_{i=1}^{n} \frac{1}{(i-1)!} \cdot \frac{B_{n-i}}{(n-i)!}\,.$$

1 Pigeonhole principle

This is one of the simplest principles in mathematics which has numerous applications.

Theorem 1. *If there are $n+1$ pigeons and n pigeonholes then at least one pigeonhole will have more than 1 pigeon.*

Proof. Let the pigeon distribution be $a_1 \le \cdots \le a_n$. We have $a_n \ge (\sum_i a_i)/n = \frac{n+1}{n} > 1$. □

This seemingly obvious theorem has many nice applications. Let us look at a few of them.

To start with, it implies that, in a group of 367 friends there are at least two whose birthdays coincide. Here, friends are the 'pigeons' and birth-dates are the 'pigeonholes'.

Example 1. Let us say that there are n users of Facebook. Show that there exist at least two people who have same number of friends. We can assume that $n \geq 3$. For small cases you can easily check that the theorem holds. Clearly the number of friends a user can have will only range from $\{0, 1, 2, \cdots, n-1\}$. There are n people and n possible number of friends, so we cannot apply the pigeonhole principle.

If someone is friends with $n-1$ people then everyone has at least one friend. That means 0 cannot appear in the possible number of friends. So there are n pigeons (users) and $n-1$ pigeonholes (number of friends), thus, at least two people will have the same number of friends.

Finally, if no one has $n-1$ friends, then again there are n pigeons (users) and $n-1$ pigeonholes (number of friends).

Exercise 2. Suppose there is an island in the shape of an equilateral triangle with side 2 km. Is it possible to assign spaces for five houses on the island, such that, no two houses are within a distance of 1 km ?

Divide the triangle into four equal regions by considering the triangle formed by the midpoints of the sides.

* Edited from Rajat Mittal's notes.

Example 2. For any n there exist at least one string of $0,1$ (decimal digits) which is divisible by n.

The difficulty lies in figuring out what are pigeons and what are pigeonholes. Suppose we consider all possible stings of $0,1$. What remainder will they leave when divided by n?

The remainder can only be $\{0, 1, \cdots, n-1\}$. So we can take any $n+1$ strings, at least 2 of them will have the same remainder when divided by n. This only implies that there are two $0,1$ strings whose difference is divisible by n. But the difference need not be a $0,1$ strings.

The question is then, can we have $n+1$ strings of $0,1$ whose difference is also an $0,1$ string? Consider the all 1's string. There are more than $n+1$ such strings and also if we subtract the smaller from the bigger one, we will get a $0,1$ string.

Note 1. The string will actually be very special. It will be all 1's followed by all 0's.

Exercise 3. Can you extend the argument to show that there are infinitely many strings of $0,1$ divisible by n.

References

1. P. J. Cameron. Combinatorics: Topics, Techniques and Algorithms. *Cambridge University Press*, 1994.
2. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.