Lecture 8: Counting

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1 Generating functions (Fibonacci)

We had obtained $\phi(t) = \sum_{i\geq 0} F_i t^i = \frac{1}{1-t-t^2}$.

We can factorize the polynomial, $1 - t - t^2 = (1-\alpha t)(1-\beta t)$, where $\alpha, \beta \in \mathbb{C}$.

Exercise 1. Show that $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$.

We can simplify the formula a bit more,

$$\phi(t) = \frac{1}{1-t-t^2} = \frac{c_1}{1-\alpha t} + \frac{c_2}{1-\beta t} = c_1(1+\alpha t + \alpha^2 t^2 + \cdots) + c_2(1 + \beta t + \beta^2 t^2 + \cdots)\,.$$

We got the second equality by putting $c_1 = \frac{1}{\sqrt{5}}\alpha$ and $c_2 = -\frac{1}{\sqrt{5}}\beta$. The final expression gives us an explicit formula for the Fibonacci sequence,

$$F_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})\,.$$

This is actually a very strong method and can be used for solving *linear recurrences* of the kind, $S_n = a_1 S_{n-1} + \cdots + a_k S_{n-k}$. Here $k, a_1, a_2, \cdots, a_k$ are constants. Suppose $\phi(t)$ is the recurrence for S_n, then by the above method,

$$\phi(t) = \frac{b_1 + b_2 t + \cdots + b_k t^{k-1}}{1 - a_1 t - a_2 t^2 - \cdots - a_k t^k} = \frac{c_1}{1-\alpha_1 t} + \cdots + \frac{c_k}{1-\alpha_k t}\,.$$

Where $\alpha_1, \cdots, \alpha_k$ are the roots (assumed distinct) of polynomial $x^k - a_1 x^{k-1} - \cdots - a_k = 0$ (replace t by $\frac{1}{x}$). This is known as the *characteristic polynomial* of the recurrence.

Exercise 2. How are the coefficients $b_1, \cdots, b_k$ or $c_1, \cdots, c_k$ determined?

Initial conditions.

So we get $S_n = c_1\alpha_1^n + \cdots + c_k \alpha_k^n$.

Note 1. For a linear recurrence if F_n and G_n are solutions then their linear combinations $aF_n + bG_n$ are also solutions. For a k term recurrence, the possible solutions are $\alpha_1^n, \cdots, \alpha_k^n$ and their linear combinations (where $\alpha_1, \cdots, \alpha_k$ are roots of the characteristic polynomial). The coefficients of the linear combination are fixed by the initial conditions.

Exercise 3. Does every polynomial over $\mathbb{C}$ has a complex root?

Fundamental theorem of algebra.

Exercise 4. What happens when the characteristic polynomial has repeated roots?

Use the power series for $(1-\alpha t)^{-d}$.

⋆ Edited from Rajat Mittal's notes.

2 Exponential generating function

What do we do when the recurrence is *non-linear* ? We will now see some related methods.

A permutation is called an *involution* if all cycles[1] in the permutation are of length 1 or 2. We are interested in counting the total number of involutions of $\{1, 2, \cdots, n\}$, call that $I(n)$. Eg. $I(0) = I(1) = 1$, $I(2) = 2$.

There can be two cases.

1. The number n maps to itself. This case will give rise to $I(n-1)$ involutions.
2. The number n maps to another number i. There are $n-1$ choices of i and then we can pick an involution for remaining $n-2$ numbers in $I(n-2)$ ways.

By this argument, we get a simple recurrence,

$$I(n) = I(n-1) + (n-1)I(n-2)\,.$$

Exercise 5. Why is this not a linear recurrence?

Since the coefficient of $I(n-2)$ is not a constant, we cannot apply the usual approach of generating functions. Even without getting an explicit formula for $I(n)$, the recurrence can give us some information about the quantity.

Theorem 1. *For $n \geq 2$, the number $I(n)$ is even and greater than $\sqrt{n!}$.*

Proof. Both the statements can be proven using induction.

Exercise 6. What will be the base case and the induction step ?

Check that $I(2), I(3)$ are both even. Verify that $1 + \sqrt{n-1} \geq \sqrt{n}$, for $n \geq 1$.

□

In the case of involutions, the regular generating function will not be of much help. We define *exponential generating function* for the sequence $I(n)$ to be,

$$\theta(t) := \sum_{k \geq 0} \frac{I(k)t^k}{k!}\,.$$

Exercise 7. Why is it called an *exponential* generating function?

Put $I(k) = 1$, for all k, and we get the series for the exponential function e^t.

Note 2. It is merely the generating function of $I(k)/k!$.

We can actually come up with a closed form solution for the exponential generating function of involutions. We will differentiate the function $\theta(t)$,

$$\frac{d}{dt}\theta(t) = \sum_{k \geq 1} \frac{I(k)t^{k-1}}{(k-1)!} \qquad \text{(formal differentiation)} \qquad (1)$$

$$= \sum_{k \geq 1} \frac{I(k-1)t^{k-1}}{(k-1)!} + \sum_{k \geq 1} \frac{(k-1) \cdot I(k-2) \cdot t^{k-1}}{(k-1)!} \qquad \text{(recurrence relation)} \qquad (2)$$

$$= \theta(t) + t \sum_{k \geq 2} \frac{I(k-2)t^{k-2}}{(k-2)!} \qquad \text{(second term's first entry is zero)} \qquad (3)$$

$$= \theta(t) + t\theta(t)\,. \qquad (4)$$

[1] Every permutation can be decomposed into cycles. Eg. (12)(3)(45)(67)(8) is an involution. Permutation (123) is not an involution.

This transforms into the differential equation,

$$\frac{d}{dt} \log \theta(t) = 1 + t\,.$$

We can solve this, as,

$$\theta(t) = e^{t+\frac{t^2}{2}+c}\,.$$

Comparing the constant coefficient, we get $c = 0$ (since $I(0) = 1$). Thus,

$$\theta(t) = e^{t+\frac{t^2}{2}}\,.$$

Note 3. Again we should notice that the power series we are considering are not shown to be well behaved (convergence etc.). But there is a justification for being able to differentiate and do other *formal* operations on them; the details are outside the scope of this course.

References

1. P. J. Cameron. Combinatorics: Topics, Techniques and Algorithms. *Cambridge University Press*, 1994.
2. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.