Lecture 7: Counting

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1 Generating functions

If one wants to find a "closed form" expression from the recurrence of F_n, one forms a *formal power series* and studies it.

The power-series $\phi(t) = \sum_{i \geq 0} F_i t^i$ is called the *generating function* for sequence F_i.

Exercise 1. Why is it not called a polynomial but a power-series?

The expression is bounded on the left-end but unbounded on the right-end.

We will be doing additions, multiplications and other operations on these power-series without worrying about the notion of convergence (or avoid substituting any real value to t). These are known as *formal power series*. The justification for not worrying about the convergence is outside the scope of this course[1].

What can we do with the generating function? Say, we continue studying the Fibonacci sequence. If we multiply it with t, we get,

$$t\phi(t) = \sum_{i \geq 0} F_i t^{i+1} = \sum_{i \geq 1} F_{i-1} t^i \,.$$

Notice how we changed the summation index. Convince yourself that it works. Multiplying it again by t,

$$t^2\phi(t) = \sum_{i \geq 0} F_i t^{i+2} = \sum_{i \geq 2} F_{i-2} t^i \,.$$

Since $F_{i-1} + F_{i-2} = F_i$ (for $i \geq 2$), adding the above two equations will give us the original generating function. Almost (why?) !!

$$\phi(t) - 1 = t\phi(t) + t^2\phi(t) \,.$$

All the coefficients for terms higher than t^2 agree. The constant term 1 appears by comparing the constant coefficient and the coefficient of t. This is where we use initial conditions !! We get the formula for $\phi(t)$,

$$\phi(t) = \frac{1}{1 - t - t^2} .$$

We can factorize the polynomial, $1 - t - t^2 = (1 - \alpha t)(1 - \beta t)$, where $\alpha, \beta \in \mathbb{R}$.

Exercise 2. Show that $\alpha = \frac{1+\sqrt{5}}{2}, \beta = \frac{1-\sqrt{5}}{2}$.

We can simplify the formula a bit more,

$$\phi(t) = \frac{1}{1 - t - t^2} = \frac{c_1}{1 - \alpha t} + \frac{c_2}{1 - \beta t} = c_1(1 + \alpha t + \alpha^2 t^2 + \cdots) + c_2(1 + \beta t + \beta^2 t^2 + \cdots) \,.$$

We got the second equality by putting $c_1 = \frac{1}{\sqrt{5}}\alpha$ and $c_2 = -\frac{1}{\sqrt{5}}\beta$. The final expression gives us an explicit formula for the Fibonacci sequence,

$$F_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1}) \,.$$

[*] Edited from Rajat Mittal's notes.

[1] Formal power series in the variable t form a ring– $\mathbb{Z}[[t]]$. So, it is a well-defined mathematical object.

References

1. P. J. Cameron. Combinatorics: Topics, Techniques and Algorithms. *Cambridge University Press*, 1994.
2. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.