Lecture 6: Counting

Nitin Saxena *

IIT Kanpur

Suppose there is a universe U with subsets $A_1, A_2, \cdots, A_n$.

We are given their intersections $A_I := \cap_{i \in I} A_i$, for every subset $I \subseteq [n]$. Then the number of elements not in any of the sets A_i is given by,

$$\left| U - \bigcup_{i \in [n]} A_i \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \cdot |A_I| \ .$$

Note 1. $A_\emptyset$ is the universe U itself. Why?

This is called the *principle of inclusion and exclusion.*

Exercise 1. Why is it called inclusion-exclusion?

Proof of the principle. We will show the equality by finding the count of an element of U in RHS. If an element $u \in U$ is not contained in any of the sets A_I then it will be counted exactly once (namely, by the term $|A_\emptyset|$).

So we only need to show that every other element is counted 0 times (overall). Suppose an element $u \in U$ is contained in A_j's for every $j \in J$, where J is the maximal nonempty subset of $[n]$. Then, the number of times u gets counted in RHS is,

$$c_J := \sum_{I \subseteq J} (-1)^{|I|} \cdot 1 \, .$$

Suppose $|J| = k$. We will be done if we can show that:

Exercise 2. $c_J = \sum_{i=0}^{k} (-1)^i \binom{k}{i} = 0$.

Consider the binomial expansion of $(1-1)^k$.

This proves that every element of U is counted exactly once if it is not in any of the A_i's, and not counted otherwise. This proves the inclusion-exclusion principle. □

Example 1. Derangements: Suppose we have n letters and n envelopes with one envelope marked for one particular letter. In how many ways could you place letter in envelopes (one letter goes to exactly one envelope), s.t., no letter goes to the correct envelope?

This is the standard application of inclusion-exclusion and known as *derangements.* There are $n!$ ways to put letters into envelopes and that is our universe U. Suppose A_i is the set of ways when letter i goes to its *correct* envelope. Hence, we are interested in $\left| U - \cup_{i \in [n]} A_i \right|$.

To apply the inclusion-exclusion formula, we need to calculate A_I. After placing $|I|$ letters in the correct position, we have $(n - |I|)!$ ways to place remaining letters. There are $\binom{n}{i}$ subsets of size i. Then the number of derangements are,

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}.$$

* Edited from Rajat Mittal's notes.

1 Recurrence relations

Recursion is a very helpful tool to solve problems in computer science and mathematics. We will look at recursion as an aid to counting.

Suppose rabbit population needs to be introduced to an island. A pair of rabbits does not breed in its first month, and produces a pair of offspring in each subsequent month (assume that there are no deaths). Starting with one newborn pair, what will be the number of pairs after n months ?

Say, we denote the number of pairs in the n^{th} month by F_n ($F_0 := 1, F_1 := 1$). There will be two kinds of rabbit-pairs making up F_n: new-born (≤ 1 month old) and older. F_{n-2} will be the new-born pairs, while F_{n-1} would be the older ones. So,

$$F_n = F_{n-1} + F_{n-2} \,.$$

This is called a recurrence relation for F_n. We gave a *combinatorial argument* for its proof. With the initial condition ($F_0 = F_1 = 1$), this recurrence gives us an easy algorithmic way to compute the population in the n-th month.

Many a times, it is hard to come up with an explicit formula for a mathematical quantity, but recurrence relation gives us valuable information about it.

Exercise 3. Suppose $F_0 = F_1 = 1$. Show that,

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} - 1.$$

The above sequence is very special and is known as *Fibonacci sequence*. That is, the numbers F_n are called Fibonacci if they satisfy the recurrence $F_n = F_{n-1} + F_{n-2}$ with the initial condition $F_0 = F_1 = 1$.

Exercise 4. There are n seating positions in a line. What is the number of ways of choosing a subset with no consecutive positions?

If n is chosen then $n-1$ isn't. Thus, $F_n = F_{n-1} + F_{n-2}$.

Exercise 5. Write the recurrence relation for S_n– the number of subsets of a set with n elements.

Either n is chosen or not. Thus, $S_n = S_{n-1} + S_{n-1}$.

Exercise 6. Prove the following bounds: $1.618^{n-1} \leq F_n \leq 2^n$.

Consider the root $\alpha > 1.618$ of $X^2 - X - 1 = 0$. Prove by induction that $\alpha^{n-1} \leq F_n \leq 2^n$.

References

1. P. J. Cameron. Combinatorics: Topics, Techniques and Algorithms. *Cambridge University Press*, 1994.
2. K. H. Rosen. Discrete Mathematics and Its Applications. *McGraw-Hill*, 1999.