Lecture 1: Introduction to discrete mathematics

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1 Administrivia

This is an introductory course to the mathematics needed in CSE. The topics covered are useful in the theory, systems and application areas of computer science.

Grading. Tentatively the activities are:

Assignments (5%): These should be done independently and submitted in time.

Quizzes (25%): Quizzes of 30 minutes each. Each will be announced a few days in advance.

Mid-sem exam (30%): 2 hours as scheduled by DoAA.

End-sem exam (40%): 3 hours as scheduled by DoAA.

Optional presentation (Bonus marks): A reading project could be taken in consultation with the TAs/ instructor. The topic should be quite advanced and interesting enough. A presentation of around 30 minutes is expected in the last month (limited slots available).

Most importantly, any immoral behavior like cheating and fraud will be punished with extreme measures and without any exception. `http://www.cse.iitk.ac.in/pages/AntiCheatingPolicy.html`

This note is an introduction, and written to give a feeling about what will be covered in the course. Hence, the terms may be loosely defined. This introduction will make more and more sense as we progress through the course.

2 What is discrete mathematics?

The branch of mathematics which deals with "discrete" objects and structures is called discrete mathematics. Here by discrete set we mean that the elements are distinct and not connected. So we can say that the set has finite or countably infinite number of elements (the elements can be counted). Do not worry if these terms are unfamiliar to you, they will become clearer as the course progresses. At this point, to get the intuition, the set of natural numbers is a discrete set. But the set of real numbers are continuous.

Discrete mathematics plays a fundamental role in Computer Science and is an essential background for almost all of the advanced courses like theory of computation, compilers, databases, operating systems, algorithms and data structures etc. . One of the main reason for its importance is that the information in a computer is stored and manipulated in a discrete fashion. Computer has a finite precision.

This course will be a collection of various concepts and techniques which will help you in your future endeavors in computer science. These different topics can be broadly divided into five distinct parts related to each other. We will introduce them one by one.

2.1 Proofs

"I have had my results for a long time, but I don't know yet how to arrive at them."
– Karl Friedrich Gauss

Proofs are the way in which we arrive at the result or show that the statement is rigorous mathematically. This will be a short but very important part of the course. These are few of the questions we will worry about in the first few lectures.

* Thanks to Prof.Rajat Mittal's original notes.

– What are proofs?
– What are the different techniques used to prove mathematical statements? (Eg. *axioms, induction, implication, contradiction.*)
– Which proofs are mathematically correct?
– Can every statement be proved/ disproved?

These questions are part of a logic course. We will tackle these questions using examples. So there will be lot of examples of proofs and different techniques used to prove things in mathematics. The target would be to get a feeling of when is a proof mathematically correct and when does it have an error.

2.2 Combinatorics

"It is difficult to find a definition of combinatorics that is both concise and complete, unless we are satisfied with the statement Combinatorics is what combinatorialists do."
– W. T. Tutte

Combinatorics can be thought of as "art of counting". It concerns itself with anything enumerative and mostly asks the questions of type, how many ways are there to do some well defined operation on a set.

As Peter Cameron explains, most of the other branches in mathematics have a well-defined goal, such as prime number theorem. But Combinatorics seems to be a collection of unrelated puzzles chosen at random. Hence this field is very broad, where emphasis is given on techniques rather than results.

In the second part of this course we will look at these different techniques and their application. The kind of puzzles we will be interested in,

– For any positive integer n, prove that there exists a positive integer consisting of digits 1 and 0 only and is divisible by n.
– Calculate the number of ways to distribute n identical balls among m distinct bins.
– Count the permutations of $\{1, 2, \cdots, n\}$ in which each element is mapped to an element different from itself.
– What are the number of valid parenthesis with n '(' and n ')'? (*recurrence, generating function*)
– In how many ways can we partition the number n?
– How many elements less than n are there which are coprime to n?

2.3 Number theory

"Mathematics is the queen of the sciences and number theory is the queen of mathematics. She often condescends to render service to astronomy and other natural sciences, but in all relations she is entitled to the first rank."
– Karl Friedrich Gauss (1777-1855)

We will study some basic properties of numbers (i.e. integers). This is an area where one can ask simple questions but there may not be an easy answer. We will see some elegant, and basic, techniques in number theory. For example,

– When can a number be factored?
– How many primes are there?
– Test whether two numbers share a factor?
– Is number theory practical? (*cryptography, error-correcting codes*)

2.4 Graph theory

"The origins of graph theory are humble, even frivolous."
– N. Biggs, E. K. Lloyd, and R. J. Wilson

Graph theory, not surprisingly, is the study of "combinatorial graphs". We have written "combinatorial graphs", to make the distinction between graphs (or diagrams) of functions and the graphs we will study. The graphs in graph theory are more like the pictorial representations of networks (communication, transport etc.).

Graph theory will be the fourth part of our course, but it is inherently related to combinatorics. Many people used to consider (still consider) it as a part of combinatorics.

Combinatorial graphs are graphical representation of combinatorics and related disciplines. They are used to model many different situations involving relations, networks and other mathematical applications. We will be interested in different kind of well-defined structures inside these graphs and counting them.

Again, the questions which will interest us will look like,

– Is there a way to go from Delhi to Mumbai using every city given the connections? (*Hamiltonian path*)
– How many classrooms do we need such that no two friends are in the same room given the friendship network of a class? (*coloring*)
– Does there exist a possible pairing of girls and boys given their preferences? (*matching*)

2.5 Probability and its applications

"The 50-50-90 rule: anytime you have a 50-50 chance of getting something right, there's a 90% probability you will get it wrong."
– Andy Rooney

This is one of the subjects in which you already had some introduction. There is no need to emphasize the importance of probability in science. Just to give a glimpse, probability is useful in statistics, physics, quantum mechanics, finance, artificial intelligence/machine learning, computer science and even gambling.

We will mostly be concerned about the role of probability in computer science and mathematics. The probability theory we will consider will mostly deal with sample spaces and sets which are discrete. The kind of questions considered in this part will be,

– You roll two dices, what is the probability that the two outcomes are co-prime.
– A pick from a box gives a random fruit (out of n) with equal chance. How many picks should we make to get every fruit?
– Suppose you flip three different coins. If the two are alike then the probability of the third one being the same is $\frac{1}{2}$. Though TTT and HHH happen with probability $\frac{1}{8}$. So all three being same happens with probability $\frac{1}{4}$. Why are the two numbers different? (*conditional probability*)
– Suppose a test for the drug gives 99% true positives and 99% true negatives. What is the probability that a particular person is a drug-user, given that he tested positive? (*Bayes' rule*)

The final part of the course will start by defining basics of probability, random variables and distributions. We will briefly discuss "Probabilistic methods", an immensely successful field which gives existential proofs (in contrast to constructive proofs) in many diverse fields.

Let us see some applications of probabilistic method. Notice that the statements do not seem to have any connection with probability.

– A family of subsets of $\{1, 2, \cdots, n\}$ is called an anti-chain if no element of the family contains another element of the family. How big can an anti-chain be?
– Prove that for every $B = \{b_1, b_2, \cdots, b_n\}$ (set of non-zero integers) there is a subset A (of size $\geq \frac{1}{3}n$) which is sum-free (no two elements of A sum up to an element of A).

3 Assignment

Exercise 1. What are the other parts of discrete mathematics? Find out on internet.

Exercise 2. Where have you used probability in your life?

Exercise 3. Read the Wikipedia entry on “Combinatorics”, “Number theory” and “Graph theory”.