Extending $\zeta(z)$ to entire plane

Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

We know that if z is +ve integer,

then $\Gamma(z) = (z-1)!$

Does $\Gamma(z)$ converge for all z?

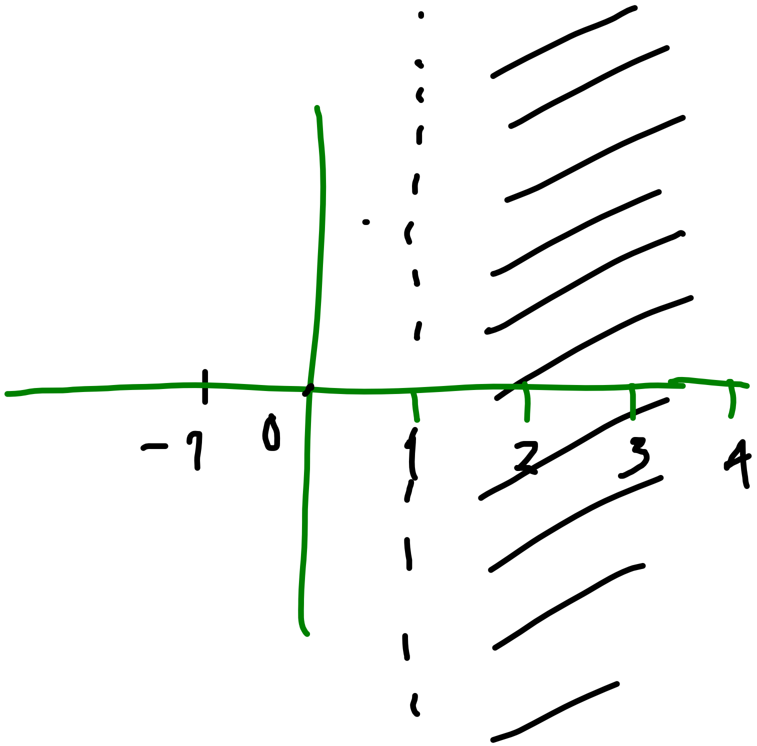
$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt$$

$$= \left[\frac{t^z}{z} e^{-t} \right]_0^\infty + \int_0^\infty \frac{t^z}{z} e^{-t} \, dt$$

$$\Rightarrow \Gamma(z) = \frac{1}{z} \Gamma(z+1)$$

So $\Gamma(0)$ does not converge!

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$



For any z, $\mathrm{Re}(z) > 1$,

$$|\Gamma(z)| \leq \left| \int_0^\infty t^{\alpha} e^{-t}\, dt \right|$$

$$= |\Gamma(\alpha+1)|$$

$$< \infty$$

Using the relation $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$,

we can define $\Gamma(z)$ for all z except

for $z = 0, -1, -2, -3, -4, \cdots$

Therefore, $\Gamma(z)$ is a meromorphic function

of $\mathbb{C}$ with simple poles at $z = 0, -1, -2, \cdots$

Consider $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$.

let $t = nu$, $n > 0$.

$$\Rightarrow \Gamma(z) = \int_0^\infty n^{z-1} u^{z-1} e^{-nu}\, n\, du$$

$$= n^z \int_0^\infty u^{z-1} e^{-nu}\, du$$

$$\Rightarrow \frac{1}{n^z} \Gamma(z) = \int_0^\infty u^{z-1} e^{-nu}\, du$$

$$\Rightarrow \left(\sum_{n>0} \frac{1}{n^z}\right)\Gamma(z) = \sum_{n>0} \int_0^\infty u^{z-1} e^{-nu}\, du$$

$$\Rightarrow \quad \zeta(z)\,\Gamma(z) = \int_0^\infty u^{z-1}\left(\sum_{n=1}^\infty e^{-nu}\right) du$$

$$= \int_0^\infty \frac{u^{z-1} e^{-u}}{1-e^{-u}}\, du$$

$$= \int_0^\infty \frac{u^{z-1}}{e^u - 1}\, du.$$

The second relationship between ζ & Γ

$$\Gamma(z/2) = \int_0^\infty t^{z/2-1} e^{-t}\, dt$$

Replace t by $\pi n^2 u$, $n = 1, 2, \ldots$.

$$\Gamma(z/2) = \int_0^\infty (\pi n^2 u)^{z/2-1} e^{-\pi n^2 u} \pi n^2 du$$

$$= \int_0^\infty (\pi n^2)^{z/2} u^{z/2-1} e^{-\pi n^2 u} du$$

$$= \int_0^\infty \pi^{z/2} n^z u^{z/2-1} e^{-\pi n^2 u} du$$

$$\Rightarrow \quad \frac{1}{n^z} \pi^{-z/2} \Gamma(z/2) = \int_0^\infty u^{z/2-1} e^{-\pi n^2 u} du$$

$$\Rightarrow \quad \zeta(z) \pi^{-z/2} \Gamma(z/2) = \int_0^\infty u^{z/2-1} \left(\sum_{n=1}^\infty e^{-\pi n^2 u} \right) du$$

let $W(u) = \sum_{n=1}^\infty e^{-\pi n^2 u}$

$$= \frac{1}{2} \left(\sum_{n=-\infty}^\infty e^{-\pi n^2 u} \right) - 1/2$$

Let $\omega(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}$.

$\boxed{\text{Aim : Relate } \omega(u) \text{ with } \omega(1/u)}$

Consider $e^{-\pi n^2 u}$. Let $f(n) = e^{-\pi n^2 u}$.

<u>Lemma</u>: The Fourier transform of $f(n)$ is $\hat{f}(m) = u^{-1/2} e^{-\pi m^2/u}$.

proof: let $f(s) = e^{-\pi s^2 u}$

Fourier transform of f is:

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(s)\, e^{-2\pi i s t}\, ds$$

$$= \int_{-\infty}^{\infty} e^{-\pi s^2 u}\, e^{-2\pi i s t}\, ds$$

$$= \int_{-\infty}^{\infty} e^{-\pi\left(s^2 u + 2ist\right)} ds$$

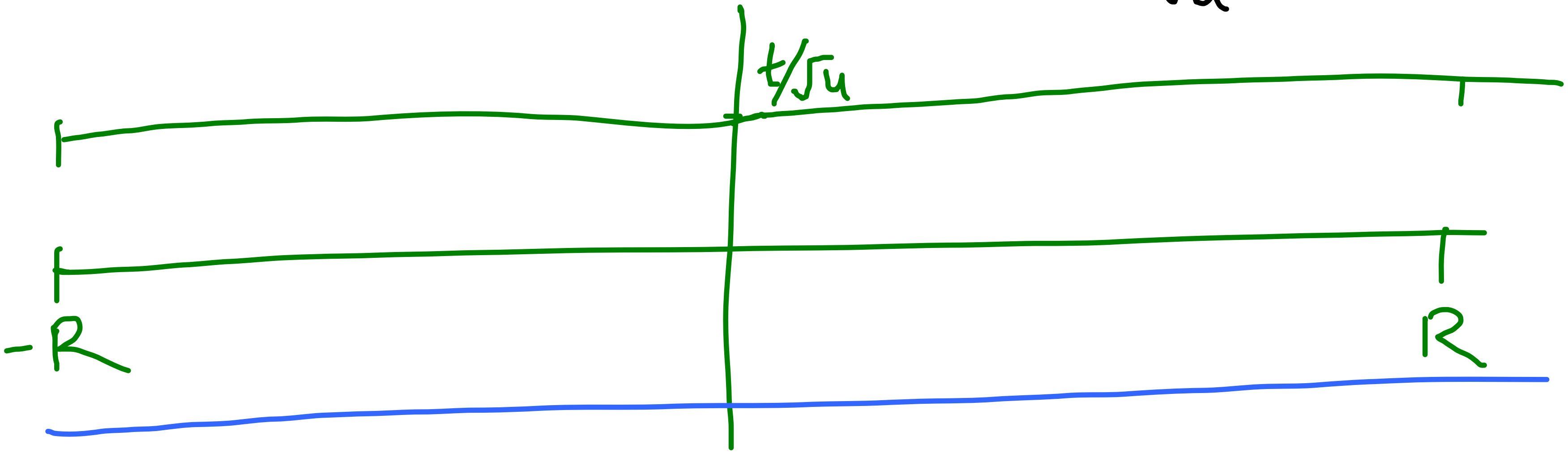
$$= \int_{-\infty}^{\infty} e^{-\pi\left(s^2 u + 2ist - t^2/u + t^2/u\right)} ds$$

$$= e^{-\pi t^2/u} \int_{-\infty}^{\infty} e^{-\pi\left(s u^{1/2} + it/u^{1/2}\right)^2} ds$$

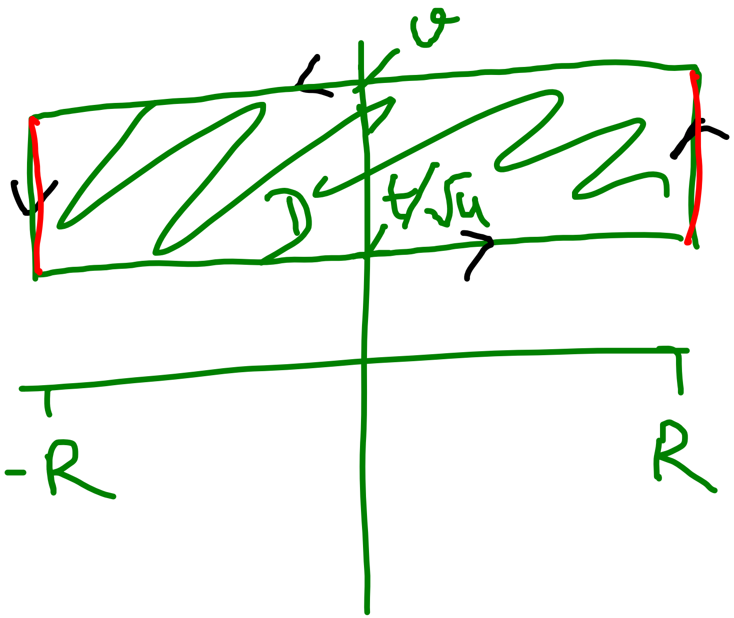
Let $z = s u^{1/2} + it/u^{1/2}$

$$= e^{-\pi t^2/u} \int_{-\infty + it/\sqrt{u}}^{+\infty + it/\sqrt{u}} e^{-\pi z^2} \frac{dz}{\sqrt{u}}$$

$$= u^{-1/2} e^{-\pi t^2/u} \int_{-\infty + it/\sqrt{u}}^{+\infty + it/\sqrt{u}} e^{-\pi z^2} \, dz$$



Consider $\displaystyle\int_{-R+it/\sqrt{u}}^{R+it/\sqrt{u}} e^{-\pi z^2}\,dz.$



$$\int_{\delta D} e^{-\pi z^2}\,dz = 0$$

$$\Rightarrow 0 = \int_{-R+it/\sqrt{u}}^{R+it/\sqrt{u}} + \int_{R+it/\sqrt{u}}^{R+iv} + \int_{R+iv}^{-R+iv} + \int_{-R+iv}^{-R+it/\sqrt{u}} e^{-\pi z^2}\,dz$$

$$\left| \int_{R+it/\sqrt{u}}^{R+iv} e^{-\pi z^2} dz \right| \leq \left| \int_{R+it/\sqrt{u}}^{R+iv} e^{-\pi R^2} dz \right|$$

$$\leq O\left(\frac{1}{e^{\pi R^2}} \right)$$

$$\to 0 \text{ as } R \to \infty .$$

Same with 4^{th} integral.

$$\Rightarrow \int_{-\infty+it/\sqrt{u}}^{+\infty+it/\sqrt{u}} e^{-\pi z^2} dz = \int_{-\infty+iv}^{+\infty+iv} e^{-\pi z^2} dz$$

Therefore,

$$\int_{-\infty+it/\sqrt{4}}^{+\infty+it/\sqrt{4}} e^{-\pi z^2}\,dz = \int_{-\infty}^{\infty} e^{-\pi y^2}\,dy$$

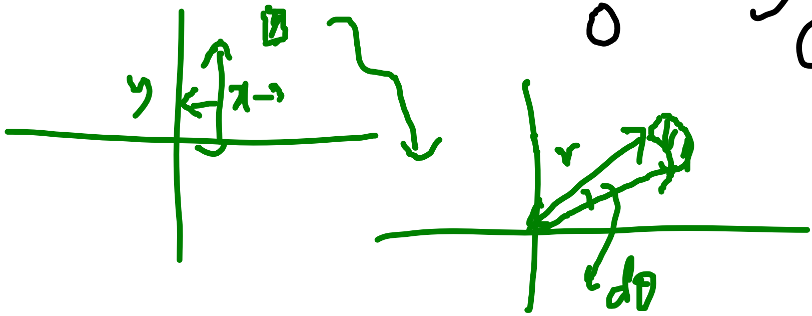
let $$I = \int_{-\infty}^{\infty} e^{-\pi y^2}\,dy$$

$$I^2 = \int_{-\infty}^{\infty} e^{-\pi y^2}\,dy \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\pi(x^2+y^2)} \, dx \, dy$$

Rewrite using polar coordinates:

$$I^2 = \int_0^{\infty} \int_0^{2\pi} e^{-\pi r^2} r \, d\theta \, dr$$



$$= 2\pi \int_0^{\infty} e^{-\pi r^2} r \, dr$$

$$= 1.$$

Therefore,

$$\hat{f}(t) = \frac{1}{u^{1/2}} e^{-\pi t^2/u}. \qquad \boxtimes$$

Coming back to:

$$\omega(u) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 u}.$$

let $$F(v) = \sum_{n=-\infty}^{\infty} e^{-\pi (n+v)^2 u}$$

Observation: $F(v) = F(v+1)$.

So, $F(\vartheta)$ is periodic with period 1.

$$\Rightarrow \quad F(\vartheta) = \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m \vartheta}.$$

Consider $\displaystyle\int_0^1 F(\vartheta) e^{-2\pi i n \vartheta} \, d\vartheta.$

$$= \int_0^1 \sum_{m=-\infty}^{\infty} c_m e^{2\pi i m \vartheta} e^{-2\pi i n \vartheta} \, d\vartheta$$

$$= \sum_{m=-\infty}^{\infty} c_m \int_0^1 e^{2\pi i (m-n) \vartheta} \, d\vartheta = c_n.$$

We have :

$$c_n = \int_0^1 F(v)\, e^{-2\pi i n v}\, dv = \underbrace{\hat{f}(n)}_{\frac{1}{\sqrt{u}} e^{-\pi n^2/u}}.$$

<u>Assignment</u> : Prove that $c_n = \hat{f}(n)$.

We have : $F(v) = \sum_{m=-\infty}^{\infty} c_m e^{-2\pi i m v}$

$$\Rightarrow \sum_{m=-\infty}^{\infty} e^{-\pi(m+v)^2 u} = \sum_{m=-\infty}^{\infty} \frac{1}{u^{1/2}} e^{-\pi m^2/u} e^{-2\pi i m v}$$

$$\Rightarrow \boxed{\sum_{m=-\infty}^{\infty} e^{-\pi m^2 u} = \sum_{m=-\infty}^{\infty} \frac{1}{u^{1/2}} e^{-\pi m^2/u}}$$

$$\underbrace{\sum_{m=-\infty}^{\infty} e^{-\pi m^2 u}}_{=\, w(u)} = \underbrace{\sum_{m=-\infty}^{\infty} \frac{1}{u^{1/2}} e^{-\pi m^2/u}}_{=\, \frac{1}{u^{1/2}} w(1/u)}$$

$$\Rightarrow \quad w(u) = u^{-1/2}\, w(1/u).$$

Since $W(u) = \frac{1}{2} w(u) - 1/2$,

$$1 + 2W(u) = u^{-1/2}\left(1 + 2W(1/u)\right)$$

$$\Rightarrow \boxed{W(1/u) = \frac{1}{2}\left[u^{1/2}\left(1 + 2W(u)\right) - 1\right]}$$

We had:

$$\zeta(z)\,\pi^{-z/2}\,\Gamma(z/2) = \int_0^\infty u^{z/2-1}\,W(u)\,du$$

$$= \int_0^1 + \int_1^\infty \left(u^{z/2-1}\,W(u)\,du\right)$$

Consider $\int_0^1 u^{z/2-1}\,W(u)\,du$, & let $u = 1/v$.

Then, $\int_0^1 u^{z/2-1}\,W(u)\,du = -\int_\infty^1 v^{1-z/2}\,W(1/v)\,\frac{dv}{v^2}$

$$= \int_1^\infty v^{-1-z/2}\, W(1/v)\, dv$$

$$= \int_1^\infty v^{-1-z/2}\, \frac{1}{2}\left[v^{1/2}\left(1+2W(v)\right) - 1 \right] dv$$

$$= \int_1^\infty \frac{1}{2} v^{-1/2-z/2}\, dv - \int_1^\infty \frac{1}{2} v^{-1-z/2}\, dv + \int_1^\infty v^{-1/2-z/2}\, W(v)\, dv$$

$$= \left[\frac{1}{2} \frac{v^{1/2-z/2}}{1/2-z/2} \right]_1^\infty - \left[\frac{1}{2} \frac{v^{-z/2}}{-z/2} \right]_1^\infty + \int_1^\infty v^{(1-z)/2}\, W(v) \frac{dv}{v}$$

$$= \frac{1}{z-1} - \frac{1}{z} + \int_1^\infty v^{(1-z)/2}\, W(v)\, \frac{dv}{v}$$

Therefore,

$$\boxed{\zeta(z)\,\pi^{-z/2}\,\Gamma(z/2) = \underbrace{\frac{1}{z(z-1)}}_{\text{poles at } z=0,1} + \int_1^\infty v^{(1-z)/2}\, w(v)\,\frac{dv}{v} + \int_1^\infty v^{z/2}\, w(v)\,\frac{dv}{v}}$$

conv. for all z

$\Rightarrow$ $\zeta(z)\,\pi^{-z/2}\,\Gamma(z/2)$ is invariant under the substitution $z \longmapsto 1-z$.

$$\Rightarrow \boxed{\zeta(z)\,\pi^{-z/2}\,\Gamma(z/2) = \zeta(1-z)\,\pi^{-(1-z)/2}\,\Gamma\!\left(\frac{1-z}{2}\right)}$$

Example:

Let $z = -\frac{1}{2} + it$

$$\Rightarrow \zeta\left(-\tfrac{1}{2}+it\right) \pi^{-(-1/2+it)/2}\, \Gamma\left(\frac{-\frac{1}{2}+it}{2}\right)$$

$$= \zeta\left(\tfrac{1}{2}-it\right) \pi^{-(1/2-it)/2}\, \Gamma\left(\frac{1/2-it}{2}\right)$$