The Function Equation for $\zeta(z)$

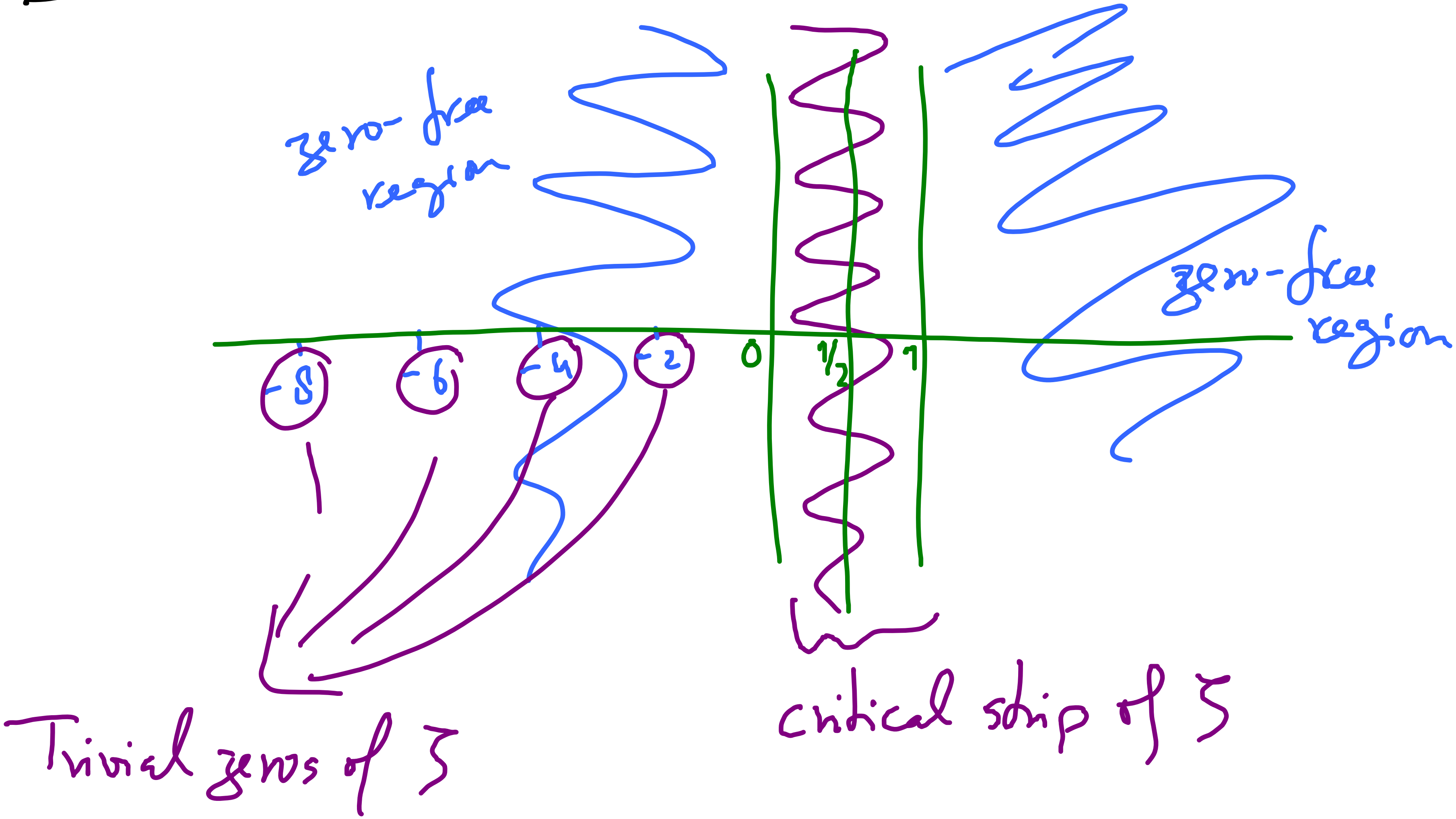
$$\zeta(z)\,\pi^{-z/2}\,\Gamma(z/2) = \zeta(1-z)\,\pi^{-(1-z)/2}\,\Gamma\left(\frac{1-z}{2}\right).$$

We also have:

$$\zeta(z)\,\pi^{-z/2}\,\Gamma(z/2) = \frac{1}{z(1-z)} + \int_1^\infty \left(v^{z/2} + v^{\frac{(1-z)}{2}}\right) w(v)\,\frac{dv}{v}$$

- $\triangleright$ $\zeta(z)$ does not have a 0 or a pole at 0.
- $\triangleright$ $\zeta(z)$ has no poles except at $z=1$
- $\triangleright$ $\zeta(z)$ has a simple zero at $z=-2m$

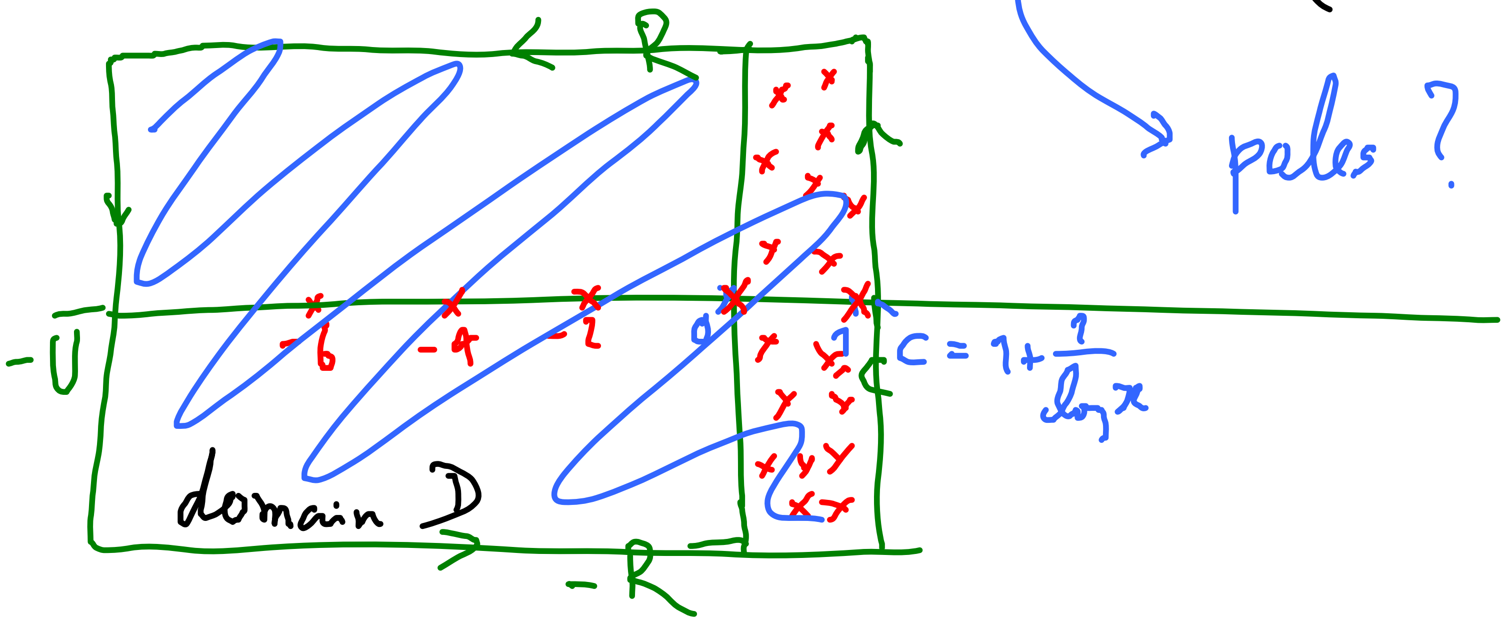
<u>Assignment</u>: Prove that $\zeta(z) \neq 0$ for $\mathrm{Re}(z) > 1$.



Going back a long time

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} -\frac{\zeta'(z)}{\zeta(z)}\cdot\frac{x^z}{z}\,dz + O\left(\frac{x\log^2 x}{R}\right)$$

poles ?



Consider $f(z)$ with pole of order k at 0.

$$\Rightarrow \quad f(z) = \frac{c}{z^k} + \cdots$$

$$\Rightarrow \quad f'(z) = -\frac{ck}{z^{k+1}} + \cdots$$

$$\Rightarrow \quad \frac{f'}{f} = \frac{-\frac{ck}{z^{k+1}} + \cdots}{\frac{c}{z^k} + \cdots}$$

$$= \frac{-\frac{ck}{z} + \cdots}{c + \cdots}$$

$\Rightarrow$ f'/f has a pole of order 1 at 0 with residue $-k$

Suppose $f(z)$ has a zero of order k at 0.

$$\Rightarrow f(z) = cz^k + \cdots$$

$$\Rightarrow f'(z) = ckz^{k-1} + \cdots$$

$$\Rightarrow f'/f = \frac{ckz^{k-1} + \cdots}{cz^k + \cdots}$$

$$= \frac{ck + \cdots}{cz + \cdots}$$

$\Rightarrow f'/f$ has a pole at 0 of order 1 & residue k.

Suppose zeros of $\zeta(z)$ in $0 \le Re(z) \le 1$ and $-R \le Im(z) \le R$ are represented by ρ.

Then,

$$\frac{1}{2\pi i}\int_{\delta D} -\frac{\zeta'}{\zeta}\frac{x^z}{z}\,dz = x - \frac{\zeta'(0)}{\zeta(0)} + \frac{x^{-2}}{2} + \frac{x^{-4}}{4} \cdots - \sum_{\substack{-R \le Im(\rho) \\ \le R}} \frac{x^\rho}{\rho}$$

$$= x - \frac{\zeta'(0)}{\zeta(0)} + \sum_{0 < 2m \le U} \frac{x^{-2m}}{2m} - \sum_{-R \le Im(\rho) \le R} x^\rho/\rho .$$

$$\frac{1}{2\pi i}\int_{\delta D} \frac{\zeta'}{\zeta}\frac{x^z}{z}\,dz = \frac{1}{2\pi i}\left[\int_{c-iR}^{c+iR} + \int_{c+iR}^{-U+iR} + \int_{-U+iR}^{-U-iR} + \int_{-U-iR}^{c-iR} \frac{\zeta'}{\zeta}\frac{x^z}{z}\,dz\right]$$

$\boxed{\text{Bounding } \zeta'/\zeta}$:

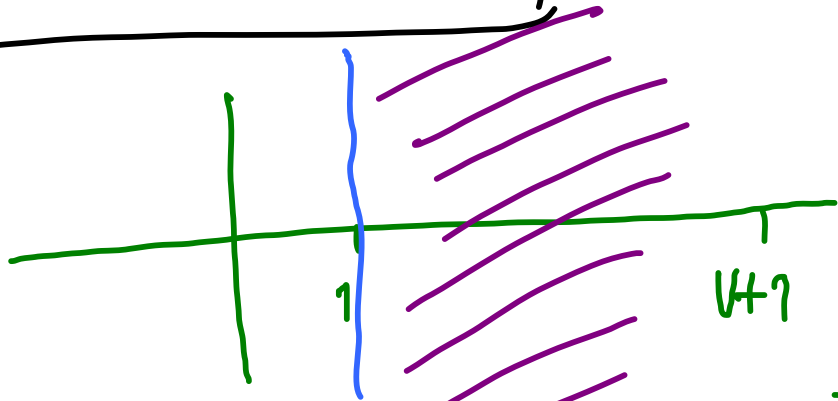
$$\pi^{-z/2}\,\Gamma(z/2)\,\zeta(z) = \pi^{-(1-z)/2}\,\Gamma\left(\frac{1-z}{2}\right)\zeta(1-z)$$

Taking log & differentiating :

$$-\frac{1}{2}\log \pi + \frac{\Gamma'(z/2)}{\Gamma(z/2)} + \frac{\zeta'(z)}{\zeta(z)} =$$

$$\frac{1}{2}\log \pi + \frac{\Gamma'\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)} + \frac{\zeta'(1-z)}{\zeta(1-z)}$$

$$\boxed{\text{For } \operatorname{Re}(z) = -U :}$$



$$|\zeta(z)| \le \sum_{n \ge 1} \frac{1}{n^{\operatorname{Re}(z)}}$$

$$\zeta'(z) = \sum_{n \ge 1} -\frac{\log n}{n^z}$$

$$\Rightarrow |\zeta'(z)| \le \sum_{n \ge 1} \frac{\log n}{n^{\operatorname{Re}(z)}}$$

$$\Rightarrow \quad \frac{\zeta'(z)}{\zeta(z)} = \frac{\Gamma'\left(\frac{1-z}{2}\right)}{\Gamma\left(\frac{1-z}{2}\right)} - \frac{\Gamma'(z/2)}{\Gamma(z/2)} + O(1)$$

$\boxed{\text{Going back to } \Gamma}$

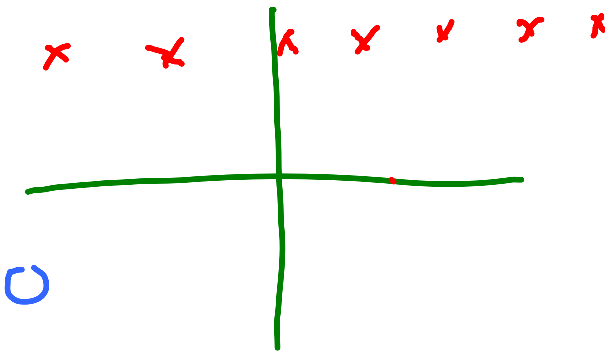
<u>Theorem</u> :
$$\Gamma(z)\,\Gamma(1-z) = \frac{\pi}{\sin \pi z}.$$

If $\Gamma(z) = 0$ then:

$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1} = 0$$

$$\Gamma(z+1) = z\,\Gamma(z) = 0$$

This is not possible.



proof:

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty t^{z-1}e^{-t}\,dt \int_0^\infty u^{-z}e^{-u}\,du$$

$$= \int_0^\infty\int_0^\infty \left(\frac{t}{u}\right)^z e^{-(t+u)}\,\frac{dt}{t}\,du$$

Let $t = uv$, $dt = u\,dv$

$$= \int_0^\infty\int_0^\infty v^z e^{-u(v+1)}\,\frac{u\,dv}{uv}\,du$$

$$= \int_0^\infty\int_0^\infty v^{z-1} e^{-u(v+1)}\,du\,dv$$

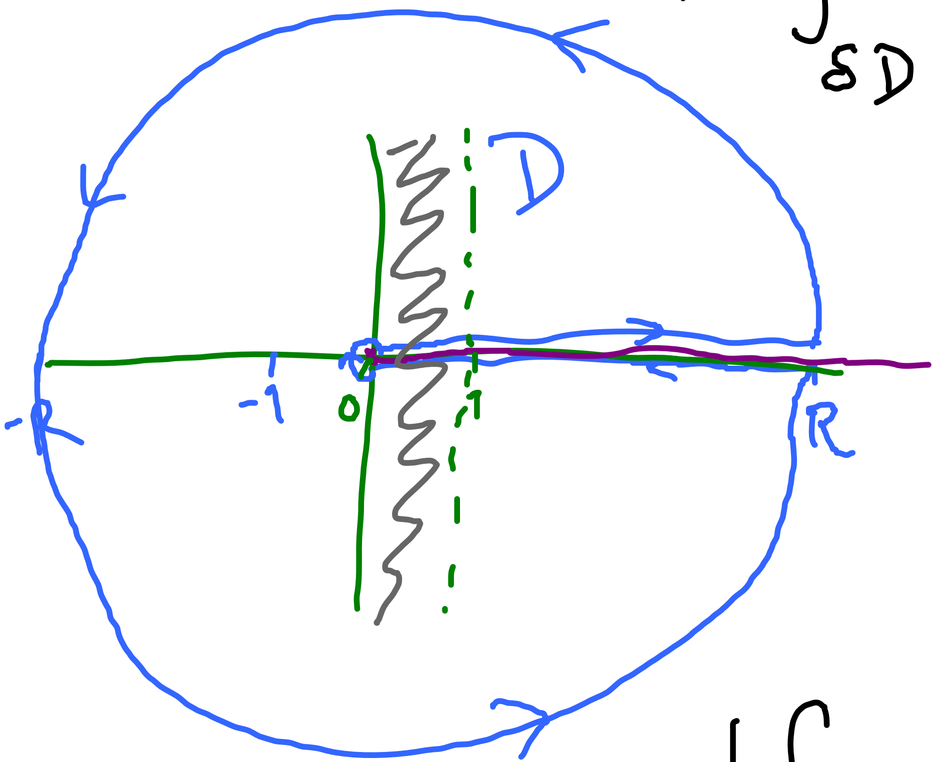
$$= \int_0^\infty v^{z-1} \left(\int_0^\infty e^{-u(v+1)} \, du \right) dv$$

$$= \int_0^\infty v^{z-1} \left[-\frac{1}{v+1} e^{-u(v+1)} \right]_0^\infty du$$

$$= \int_0^\infty \frac{v^{z-1}}{v+1} \, dv$$

Consider $\frac{1}{2\pi i}\int_{\delta D} \frac{w^{z-1}}{w+1}\, dw$

$$= (-1)^{z-1} = e^{\pi i (z-1)}.$$



$$\int_{\delta D} = \int_{\epsilon}^{R} + \int_{|w|=R} + \int_{R}^{\epsilon} + \int_{|w|=\epsilon}$$

$$\left| \int_{|w|=R} \frac{w^{z-1}}{w+1}\, dw \right| \leq \int_0^{2\pi} \frac{R^{\mathrm{Re}(z)-1}}{|w+1|} R\, d\theta$$

$$\leq \int_0^{2\pi} \frac{R^{\mathrm{Re}(z)-1}}{R-1} R \, d\theta$$

$$= O\left(R^{\mathrm{Re}(z)-1}\right) \to 0 \quad \boxed{\text{for } \mathrm{Re}(z) < 1}$$

$$\left| \int_{|w|=\epsilon} \frac{w^{z-1}}{w+1} \, dw \right| \leq \int_0^{2\pi} \frac{\epsilon^{\mathrm{Re}(z)-1}}{1-\epsilon} \epsilon \, d\theta$$

$$= O\left(\epsilon^{\mathrm{Re}(z)}\right) \to 0 \quad \boxed{\text{for } \mathrm{Re}(z) > 0}$$

$$\int_{\epsilon}^{R} \frac{w^{z-1}}{w+1}\, dw + \int_{R}^{\epsilon} \frac{w^{z-1} e^{2\pi i (z-1)}}{w e^{2\pi i} + 1}\, dw$$

$$= \int_{\epsilon}^{R} \frac{w^{z-1}}{w+1} \left(1 - e^{2\pi i (z-1)}\right) dw$$

$$= \int_{\epsilon}^{R} \frac{w^{z-1}}{w+1} \left(1 - e^{2\pi i z}\right) dw$$

Therefore:

$$\left(1-e^{2\pi i z}\right)\int_0^{\infty} \frac{v^{z-1}}{v+1}\,dv = 2\pi i\, e^{\pi i (z-1)}$$

$$\Rightarrow \Gamma(z)\,\Gamma(1-z) = \frac{2\pi i\, e^{\pi i z}}{e^{2\pi i z}-1}$$

$$= \frac{2\pi i}{e^{\pi i z}-e^{-\pi i z}}$$

$$= \frac{2\pi i}{2i \sin \pi z} = \frac{\pi}{\sin \pi z} \quad \blacksquare$$

$$\Rightarrow \quad \frac{1}{\Gamma(z)} = \frac{\sin \pi z}{\pi} \Gamma(1-z)$$

$$\Rightarrow \quad \frac{1}{|\Gamma(z)|} = e^{O(|z|)} \, |\Gamma(1-z)|$$

$$= e^{O(|z|)} \, |\Gamma(-z-k)| \, \frac{|z||z+1| \cdots}{|z+k-1|}$$

$$= e^{O(|z|)} \, |z|^{O(|z|)}$$

$$= e^{O(|z| \log |z|)}.$$

$\Rightarrow$ $1/\Gamma(z)$ is an $\boxed{\text{entire function of order 1}}$.