We know that $\frac{1}{\Gamma(z)}$ is an entire function of order 1

Therefore, $$\frac{1}{\Gamma(z)} = e^{Az+B} \cdot z \cdot \prod_{n \geq 1} \left(1+\frac{z}{n}\right) e^{-z/n}.$$

Since $\lim_{z \to 0} \frac{1}{z\Gamma(z)} = 1$, $e^{B} = 1 \Rightarrow B = 0.$

Since $\Gamma(z)=1$, $$e^{A} \cdot \prod_{n \geq 1} \left(1+\frac{1}{n}\right) e^{-1/n} = 1.$$

Consider $\prod_{1 \le n \le N} \left(1 + \frac{1}{n}\right) e^{-1/n}$.

$$\prod_{1 \le n \le N} \left(1 + \frac{1}{n}\right) e^{-1/n} = \prod_{1 \le n \le N} \frac{n+1}{n} e^{-1/n}$$

$$= e^{\sum_{1 \le n \le N} -1/n} \cdot \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \cdots \cdot \frac{N+1}{N}$$

$$= (N+1)\, e^{-\sum_{1 \le n \le N} 1/n}$$

$$= e^{\log (N+1) - \sum_{1 \le n \le N} 1/n}$$

$$= e^{\log \frac{N+1}{N} + \left(\log N - \sum_{1 \le n \le N} 1/n\right)}$$

$$= e$$

Therefore,

$$\prod_{n\geq 1}\left(1+\frac{1}{n}\right)e^{-1/n} = e^{-\gamma}$$

$$\Rightarrow \quad A = \gamma.$$

Hence:

$$\boxed{\frac{1}{\Gamma(z)} = e^{\gamma z}\cdot z\cdot\prod_{n\geq 1}\left(1+\frac{z}{n}\right)e^{-z/n}.}$$

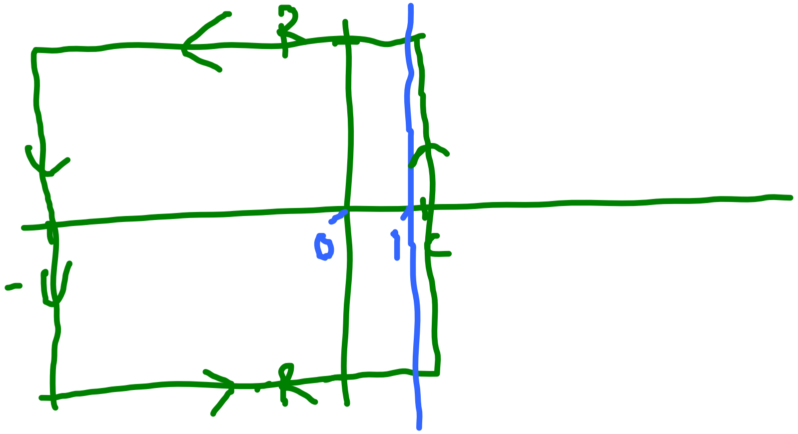
Taking log:

$$-\log \Gamma(z) = \gamma z + \log z + \sum_{n\geq 1}\left[\log\left(1+\frac{z}{n}\right) - \frac{z}{n}\right]$$

{valid when z is not on -ve real line}

$$\Rightarrow \quad -\frac{\Gamma'(z)}{\Gamma(z)} = \gamma + \frac{1}{z} + \sum_{n\geq 1}\left[\frac{\frac{1}{n}}{1+z/n} - \frac{1}{n}\right]$$

$$= \gamma + \frac{1}{z} + \sum_{n\geq 1}\left(\frac{1}{n+z} - \frac{1}{n}\right).$$



To start with, we consider $z \in [-U+iR, -U-iR]$, for U an odd integer;

For such z's, I'/I is bounded.

$$\left| \frac{I'(z)}{I(z)} \right| \leq O(1) + \left| \sum_{n\geq 1} \left(\frac{1}{z+n} - \frac{1}{n} \right) \right| .$$

Consider $\displaystyle\sum_{1 \le n \le N} \left(\frac{1}{z+n} - \frac{1}{n} \right)$.

<u>Euler-Mclaurin Formula</u>

$$\sum_{a \le n \le b} f(n) = \int_a^b f(t)\,dt + \frac{1}{2}\left(f(a)+f(b)\right) + \int_a^b f'(t)\left(t - \lfloor t \rfloor - \tfrac{1}{2}\right)dt.$$

Consider $\int_a^b f(t)\, dt$.

$$\int_k^{k+1} f(t)\cdot 1\, dt = \Big[f(t)\,(t-k-\tfrac{1}{2})\Big]_k^{k+1} - \int_k^{k+1} f'(t)\,(t-k-\tfrac{1}{2})\, dt$$

$$= \frac{1}{2}f(k+1) + \frac{1}{2}f(k) - \int_k^{k+1} f'(t)(t-k-\tfrac{1}{2})\,dt$$

$$\Rightarrow \int_a^b f(t)\, dt = \sum_{a\le n\le b} f(n) - \frac{1}{2}\big[f(a)+f(b)\big] - \int_a^b f'(t)\,(t-\lfloor t\rfloor-\tfrac{1}{2})\,dt$$

Therefore,

$$\sum_{1 \le n \le N} \frac{1}{z+n} = \int_1^N \frac{1}{z+t}\,dt + \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z+N}\right) - \int_1^N \frac{1}{(z+t)^2}\left(t - \lfloor t \rfloor - \tfrac{1}{2}\right)dt$$

$$= \log(z+N) - \log(z+1) + \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z+N}\right) - \int_1^N \frac{t - \lfloor t \rfloor - 1/2}{(z+t)^2}\,dt$$

$$\left| \int_1^N \frac{t - [t] - 1/2}{(z+t)^2} \, dt \right| \le \frac{1}{2} \left| \int_1^N \frac{dt}{(z+t)^2} \right|$$

$$= \frac{1}{2} \left| -\frac{1}{z+t} \right|_1^N$$

$$= \frac{1}{2} \left| \frac{1}{z+1} - \frac{1}{z+N} \right| .$$

Therefore,

$$\sum_{1 \le n \le N} \frac{1}{z+n} = \log(z+N) - \log(z+1) + O\left(\frac{1}{z+1} + \frac{1}{z+N} \right).$$

Therefore,

$$\left| \sum_{1 \le n \le N} \frac{1}{z+n} - \frac{1}{n} \right|$$

$$= O(1) + O\left(\left| \frac{1}{z+1} + \frac{1}{z+N} \right| \right) + \left| \log(z+N) - \log(z+1) - \log N \right|$$

$$= O(1) + O\left(\left| \frac{1}{z+1} + \frac{1}{z+N} \right| \right) + \left| \log\left(1 + \frac{z}{N}\right) - \log(z+1) \right|$$

Therefore,

$$\left|\frac{\Gamma'(z)}{\Gamma(z)}\right| = \lim_{N\to\infty}\left[\left|\log\left(1+\frac{z}{N}\right) - \log(z+1)\right| + O\left(1+\frac{1}{|z+1|}+\frac{1}{|z+N|}\right)\right]$$

$$= \left|\log(z+1)\right| + O(1),$$

$$\left\{\text{for } z \in [-U+iR, -U-iR]\right\}.$$

__Recall__:

We are trying to estimate $\left| \int_{-U+iR}^{-U-iR} \frac{\zeta'(z)}{\zeta(z)} \cdot \frac{x^z}{z} \, dz \right|$

$$\left| \frac{\zeta'(z)}{\zeta(z)} \right| = \left| \frac{\Gamma'(z/2)}{\Gamma(z/2)} \right| + \left| \frac{\Gamma'(\frac{1-z}{2})}{\Gamma(\frac{1-z}{2})} \right| + \left| \frac{\zeta'(1-z)}{\zeta(1-z)} \right| + O(1).$$

For $z \in [-U-iR, -U+iR]$:

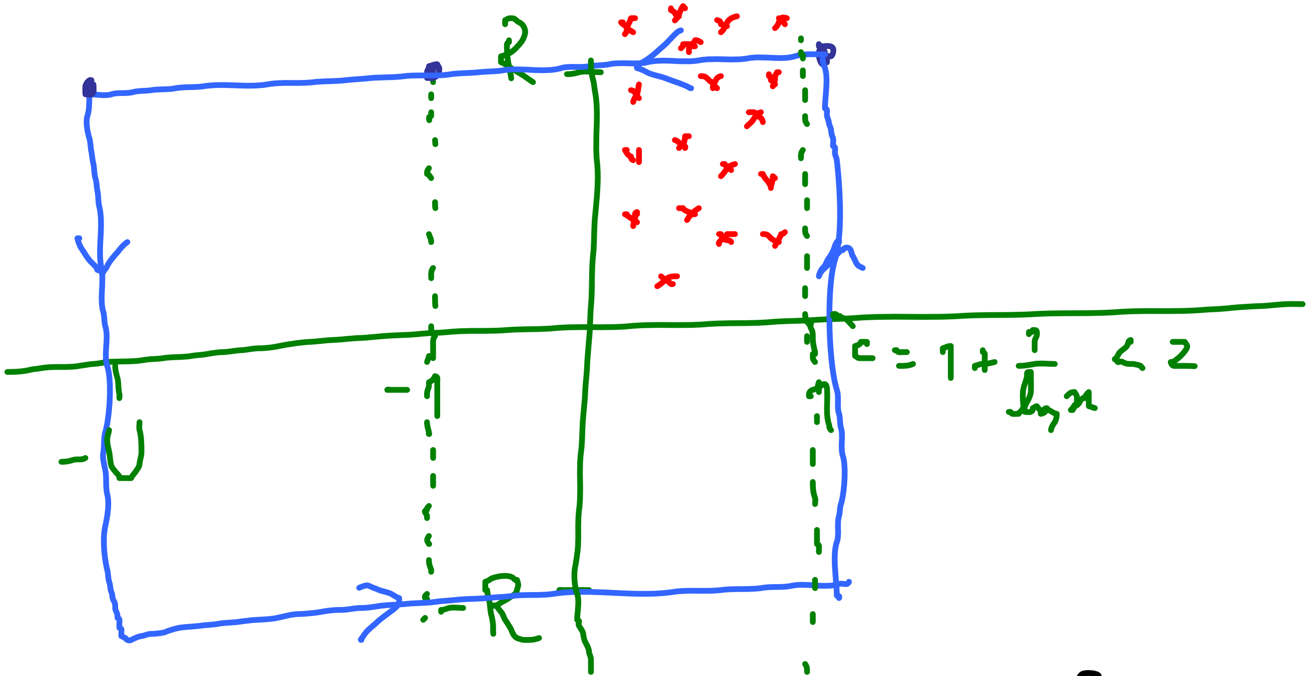
$$\left| \frac{\zeta'(1-z)}{\zeta(1-z)} \right| = O(1).$$

Therefore,

$$\left|\frac{\zeta'(z)}{\zeta(z)}\right| = \left|\log\left(1+z/2\right)\right| + \left|\log\left(1+\frac{1-z}{2}\right)\right| + O(1)$$
$$= O\left(\log|z|\right).$$

Hence :

$$\left|\int_{-U+iR}^{-U-iR} \frac{\zeta'(z)}{\zeta(z)} \cdot \frac{x^z}{z}\, dz\right| \leq \left|\int_{-U-iR}^{-U+iR} O\left(\log|z|\right) \frac{x^{-U}}{U}\, dz\right|$$
$$= O\left(\frac{R \log(R^2+U^2)}{U\, x^U}\right).$$



$$\left| \int_{-1-iR}^{-U-iR} \frac{\zeta'(z)}{\zeta(z)} \cdot \frac{x^z}{z}\, dz \right| \leq \left| \int_{+1-iR}^{+U-iR} O(\log |z|) \frac{1}{R\, x^{\mathrm{Re}(z)}}\, dz \right|$$

$$= O\left(\left| \int_1^U \frac{\log(R^2+t^2)}{x^t (R^2+t^2)^{1/2}}\, dt \right| \right)$$

$$= O\left(\left| \int_1^U \frac{dt}{x^t (R^2+t^2)^{1/2-\epsilon}} \right| \right)$$

$$= O\left(\left| \int_1^U \frac{dt}{x^t \, R^{1-\epsilon}} \right| \right)$$

$$= O\left(\left| \frac{1}{\ln x \, x^t \, R^{1-\epsilon}} \right|_1^U \right)$$

$$= O\left(\frac{1}{x \ln x \, R^{1-\epsilon}} + \frac{1}{\ln x \, R^{1-\epsilon} x^U} \right)$$

$$= O\left(\frac{1}{x \ln x \, R^{1-\epsilon}} \right).$$

Now consider:

$$\left| \int_{c+iR}^{-1+iR} \frac{\zeta'(z)}{\zeta(z)} \frac{x^z}{z} \, dz \right| .$$

We cannot use the functional equation to bound $\left| \frac{\zeta'(z)}{\zeta(z)} \right|$ in the above region as $\left| \frac{\zeta'(1-z)}{\zeta(1-z)} \right|$ is not necessarily bounded.

The function $\xi(z)$

$$\boxed{\xi(z) = z(z-1)\,\pi^{-z/2}\,\Gamma(z/2)\,\zeta(z).}$$

Recall:

$$\pi^{-z/2}\,\Gamma(z/2)\,\zeta(z) = \frac{1}{z(z-1)} + \int_1^\infty \left(t^{z/2} + t^{(1-z)/2}\right) W(t)\,\frac{dt}{t}$$

Therefore,

$$\xi(z) = 1 + z(z-1)\int_1^\infty \left(t^{z/2} + t^{(1-z)/2}\right) W(t)\,\frac{dt}{t}.$$

<u>Observation</u>: (1) $\xi(z)$ is an entire function!

(2) $\xi(z) = \xi(1-z)$.

(3) Zeros of $\xi(z)$ are precisely non-trivial zeros of $\zeta(z)$.

If $-1 < a < 1$ then

$$\int_{|z|=1} \frac{2\,dz}{z^2+2aiz-1} = 2\int_{|z|=1} \frac{dz}{(z+ai+\sqrt{1-a^2})(z+ai-\sqrt{1-a^2})}$$

$$= 2\cdot 2\pi i \left[\frac{1}{-2\sqrt{1-a^2}} + \frac{1}{2\sqrt{1-a^2}}\right]$$

$$= 0$$

If $a > 1$ then

$$\int_{|z|=1} \frac{2\,dz}{z^2+2aiz-1} = 4\pi i \left[-\frac{1}{2\sqrt{1-a^2}}\right]$$

$$= -\frac{2\pi i}{\sqrt{1-a^2}} = +\frac{2\pi}{\sqrt{a^2-1}}$$