Theory of Entire Functions of Order ?

Function f is entire if it is analytic on $\mathbb{C}$.

Function f is entire function of order k if for all $z \in \mathbb{C}$:

$$|f(z)| = e^{O(|z|^{k+\epsilon})}$$

for some small $\epsilon > 0$.

<u>Theorem</u>: If f is an entire function of order 1 and has no zeros, then $f(z) = e^{Az+B}$.

proof: let $g(z) = \log f(z) - \log f(0)$.

Function g is also entire, and $g(0) = 0$.

$$\operatorname{Re}(g(z)) = \log|f(z)| - \log|f(0)|$$

$$= O(|z|^{1+\epsilon}).$$

<u>Lemma</u>: If $\mathrm{Re}(g(z)) \le M$ for $|z| \le 2R$, then
$|g(z)| \le 2M$ for $|z| \le R$.

<u>proof of Lemma</u>: $\hat{g}(z) = \dfrac{2R\, g(z)}{z\,(2M - g(z))}$.

$\hat{g}(z)$ is analytic in disk $|z| \le 2R$.

$$\Rightarrow |\hat{g}(w)| \le |\hat{g}(z)|, \quad |z| = 2R$$

$$= \frac{2R\, |g(z)|}{|z|\, |2M - g(z)|}$$

$$= \frac{\sqrt{\alpha^2 + \beta^2}}{\sqrt{(2M - \alpha)^2 + \beta^2}} \le 1. \quad (g(z) = \alpha + i\beta)$$

$$2Rg(z) = \hat{g}(z) \cdot z(2M - g(z))$$

$$\Rightarrow \left(2R + z\hat{g}(z)\right) g(z) = 2Mz\hat{g}(z)$$

$$\Rightarrow g(z) = \frac{2Mz\hat{g}(z)}{2R + z\hat{g}(z)} = \frac{Mz}{R} \frac{\hat{g}(z)}{1 + \frac{z}{2R}\hat{g}(z)}$$

For $|z| = R$, $$|g(z)| = \frac{M|z|}{R} \frac{|\hat{g}(z)|}{\left|1 + \frac{z}{2R}\hat{g}(z)\right|}$$

$$\leq \frac{M}{\left|1 + \frac{z}{2R}\hat{g}(z)\right|} \leq 2M. \quad \blacksquare$$

Therefore, when $Re(g) = O(|z|^{1+\epsilon})$,

then $|g(z)| = O(|z|^{1+\epsilon})$.

Since g is analytic over $\mathbb{C}$,

$$g(z) = \sum_{m \geq 0} c_m z^m$$

$$\Rightarrow \quad g(re^{i\theta}) = \sum_{m \geq 0} c_m r^m e^{i\theta m}$$

$$\Rightarrow \quad \int_0^{2\pi} g(re^{i\theta}) e^{-ik\theta} d\theta = \sum_{m \geq 0} c_m r^m \int_0^{2\pi} e^{i\theta(m-k)} d\theta$$

$$\Rightarrow \quad c_k r^k 2\pi = \int_0^{2\pi} g(re^{i\theta}) e^{-ik\theta} \, d\theta$$

$$\Rightarrow \quad |c_k| r^k = \int_0^{2\pi} O\left(|g(re^{i\theta})|\right) d\theta$$

$$= O(r^{1+\epsilon})$$

$\Rightarrow$ $c_k = 0$ for $k > 1$.

$$\Rightarrow \quad g(z) = c_0 + c_1 z$$

Since $g(0) = 0$, $g(z) = c_1 z$.

$$\Rightarrow \quad f(z) = f(0) e^{g(z)} = f(0) e^{c_1 z} \qquad \boxtimes$$

Theorem: Let f be an entire function of order 1, and with finitely many zeros $z_1, z_2, \dots, z_k$.

Then,
$$f(z) = \left(\prod_{i=1}^{k} (z - z_i) \right) e^{Az+B}.$$

Theorem: Let f be an entire function of order 1, $f(0) \neq 0$, and with infinitely many zeros $z_1, z_2, z_3, \dots$

Then,
$$f(z) = e^{Az+B} \prod_{i \geq 1} \left(1 - \frac{z}{z_i}\right) e^{z/z_i}.$$

<u>proof</u>: We first estimate the number of roots of f in the disk $|z| \le R$.

<u>Lemma</u>: Let $z_1, z_2, \dots, z_t$ be roots of f inside the disk $|z| \le R$. Then:

$$\prod_{i=1}^{t} \frac{R}{|z_i|} \le e^{O(R^{1+\epsilon})}.$$

<u>proof of Lemma</u>: Let $g(z) = \left[\prod_{i=1}^{t} \frac{(R^2 - z\bar{z}_i)}{R(z - z_i)}\right] \cdot f(z)$.

for $|z| = R$, $|g(z)| = \left(\prod_{i=1}^{t} \frac{|R^2 - z\bar{z}_i|}{R|z - z_i|}\right) |f(z)|$

$$= \left(\prod_{i=1}^{t} \frac{|z\bar{z} - \bar{z}_i z|}{R\,|z - z_i|} \right) |f(z)|$$

$$= \left(\prod_{i=1}^{t} \frac{|z|\,|\bar{z} - \bar{z}_i|}{R\,|z - z_i|} \right) |f(z)|$$

$$= |f(z)| .$$

Therefore: $$|g(0)| \leq \max_{|z|=R} \{|g(z)|\}$$

$$= \max_{|z|=R} \{|f(z)|\}$$

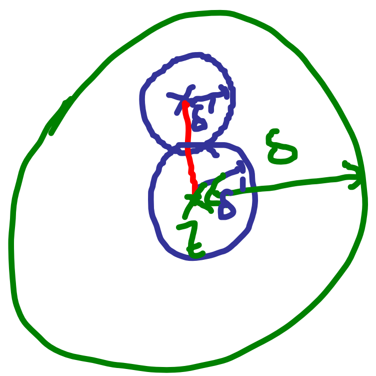
$$= e^{O(R^{1+\epsilon})}$$

Let f be an analytic function on D..

Lemma: For any $z \in D$, there is a small δ such that:

(i) if $f(z) = 0$, then the disk $|z - w| < \delta$ has only one zero

(ii) if $f(z) \neq 0$, then the disk $|z - w| < \delta$ has no zeros.



$$g(0) = \left[\prod_{i=1}^{t} \frac{R^2}{R(-z_i)}\right] f(0)$$

$$= (-1)^t \left(\prod_{i=1}^{t} \frac{R}{z_i}\right) f(0).$$

$$\Rightarrow |g(0)| = \left(\prod_{i=1}^{t} \frac{R}{|z_i|}\right) |f(0)|.$$

$$\Rightarrow \prod_{i=1}^{t} \frac{R}{|z_i|} = e^{O(R^{1+\epsilon})}. \qquad \boxtimes$$

Let l of these z_i's have absolute value $\leq R/2$.

$$\Rightarrow \prod_{i=1}^{l} \frac{R}{|z_i|} \leq \prod_{i=1}^{t} \frac{R}{|z_i|} = e^{O(R^{1+\epsilon})}$$

$$\Rightarrow \quad 2^{\ell} \leq \prod_{i=1}^{\ell} \frac{R}{|z_i|} = e^{O(R^{1+\epsilon})}$$

$$\Rightarrow \quad \ell = O(R^{1+\epsilon}).$$

Let $n(R)$ the number of zeros of f in the disk $|z| < R$, then $\boxed{n(R) = O(R^{1+\epsilon}).}$

<u>Lemma</u>: For any $\delta > \epsilon$, $\sum_{i \geq 1} \frac{1}{|z_i|^{1+\delta}} < \infty$.

proof: $$\sum_{i\geq 1} \frac{1}{|z_i|^{1+\delta}} = \sum_{k \geq 1} \sum_{2^{k-1} \leq |z_i| < 2^k} \frac{1}{|z_i|^{1+\delta}}$$

$$\leq \sum_{k\geq 1} \sum_{2^{k-1}\leq |z_i| < 2^k} \frac{1}{2^{(k-1)(1+\delta)}}$$

$$\leq \sum_{k\geq 1} \frac{c\,2^{k(1+\epsilon)}}{2^{(k-1)(1+\delta)}}$$

$$= 2^{1+\delta}\cdot c \cdot \sum_{k\geq 1} \frac{2^{k(1+\epsilon)}}{2^{k(1+\delta)}}$$

$$= c\,2^{1+\delta} \sum_{k\geq 1} \frac{1}{2^{k(\delta-\epsilon)}}$$

$$= c\,2^{1+\delta}\cdot \frac{1}{2^{(\delta-\epsilon)}}\cdot \frac{1}{1-\frac{1}{2^{\delta-\epsilon}}} = O(1).$$

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Back to proof of Theorem

Consider $\displaystyle\prod_{i\geq 1}\left(1-\frac{z}{z_i}\right)e^{z/z_i}$ for $|z|=R$.

$$\prod_{i\geq 1} = \prod_{|z_i|\leq 2R}(\quad)\cdot\prod_{|z_i|>2R}(\quad)$$

$$\prod_{|z_i|>2R}\left(1-z/z_i\right)e^{z/z_i} = \prod_{|z_i|>2R} e^{\log(1-z/z_i)+z/z_i}$$

$$= \prod_{|z_i|>2R} e^{z/z_i-\sum_{j\geq 1}\frac{(z/z_i)^j}{j}}$$

$$= \prod_{|z_i| > 2R} e^{z/z_i - \sum_{j \geq 1} \frac{z^j}{j z_i^j}}$$

$$= \prod_{|z_i| > 2R} e^{- \sum_{j \geq 2} \frac{z^j}{j z_i^j}}$$

Therefore,

$$\left| \prod_{|z_i| > 2R} (\quad) \right| \leq \prod_{|z_i| > 2R} e^{\sum_{j \geq 2} \frac{|z|^j}{j |z_i|^j}}$$

$$\leq \prod_{|z_i| > 2R} e^{\frac{|z|^2}{|z_i|^2} \sum_{j \geq 2} \frac{|z|^{j-2}}{|z_i|^{j-2}}}$$

$$\leq \prod_{|z_i|>2R} e^{2|z|^2/|z_i|^2}$$

$$= e^{\sum_{|z_i|>2R} \frac{2|z|^2}{|z_i|^2}}$$

$$\leq e^{2|z|^2 \sum_{i\geq 1} \frac{1}{|z_i|^2}}$$

$$= e^{O(|z|^2)}.$$

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$$\leq e^{\sum_{|z_i|>2R} \frac{2|z|^{1+\delta}}{|z_i|^{1+\delta}}}$$

$$\leq e^{2|z|^{1+\delta} \sum_{i\geq 1} \frac{1}{|z_i|^{1+\delta}}} = e^{O(|z|^{1+\delta})}.$$

Therefore, $\prod_{i \geq 1} \left(1 - \frac{z}{z_i}\right) e^{z/z_i}$ is an entire function with same set of zeros as f.

Consider $\dfrac{f(z)}{\prod_{i \geq 1} \left(1 - z/z_i\right) e^{z/z_i}}$.

We will show that $\dfrac{1}{\left|\prod_{i \geq 1} \left(1 - z/z_i\right) e^{z/z_i}\right|} \leq e^{O(|z|^{1+\epsilon'})}$.

$$\frac{1}{\prod_{i\geq 1}(\quad)} = \frac{1}{\prod_{|z_i|<|z|/2}(\quad)} \cdot \frac{1}{\prod_{|z|/2\leq|z_i|\leq 2|z|}(\quad)} \cdot \frac{1}{\prod_{|z_i|>2|z|}(\quad)}$$

$$\leq e^{O(|z|^{1+\delta})}$$

$$\frac{1}{\left|\prod_{|z_i|<|z|/2}\left(1-\frac{z}{z_i}\right)e^{z/z_i}\right|} = \prod_{|z_i|<|z|/2} \frac{|z_i|}{|z-z_i|\left|e^{z/z_i}\right|}$$

$$\leq \prod_{|z_i|<|z|/2} \frac{|z|/2}{|z|/2}\, e^{\frac{|z|}{|z_i|}}$$

$$= e^{\sum_{|z_i|<|z|/2}\frac{|z|}{|z_i|}}$$

$$= e^{\sum_{|z_i| < |z|/2} \frac{|z|}{|z_i|}}$$

$$\leq e^{\sum_{|z_i| < |z|/2} \frac{|z|^{1+\delta}}{|z_i|^{1+\delta}}}$$

$$= e^{O\left(|z|^{1+\delta}\right)}.$$

$$\left| \prod_{|z|/2 < |z_i| \leq 2|z|} \frac{1}{\left(1 - z/z_i\right) e^{z/z_i}} \right| \leq \prod_{|z|/2 < |z_i| \leq 2|z|} \frac{|z_i|}{|z - z_i|} e^{|z|/|z_i|}$$

$$\leq \prod_{|z|/2 < |z_i| \leq 2|z|} \frac{2|z| e^2}{|z - z_i|}$$

Choose a z such that $|z| \notin \bigcup_{i \geq 1} \left\{ \left[-\frac{1}{|z_i|^2} + |z_i|, \frac{1}{|z_i|^2} + |z_i| \right] \right\}$

Total values ommitted $\leq \sum_{i \geq 1} \frac{2}{|z_i|^2} = O(1)$.

For such a z;

$$\prod_{|z|/2 < |z_i| \leq 2|z|} \frac{2|z| e^2}{|z - z_i|} \leq \prod_{|z|/2 < |z_i| \leq 2|z|} 2|z| e^2 |z_i|^2$$

$$\leq \prod_{|z|/2 < |z_i| \leq 2|z|} 8 e^2 |z|^3$$

$$\leq \left[8 e^2 |z|^3 \right]^{O(|z|^{1+\delta})}$$

$$\leq e^{O(|z|^{1+\delta'})}.$$

Therefore, $\dfrac{f(z)}{\prod_{i\geq 1}(1-z/z_i)e^{z/z_i}}$ is an entire function of order 1 with no zeros.

Hence, $$\boxed{f(z) = e^{Az+B}\prod_{i\geq 1}(1-z/z_i)e^{z/z_i}.}$$

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