Counting Primes

Question: How many prime numbers are there is interval $[1, x]$?

Conjecture [Gauss]: $\pi(x) \to \dfrac{x}{\log x}$ as $x \to \infty$

where $\pi(x)$ = number of primes in interval $[1, x]$.

$$\pi(x) = \sum_{n=1}^{x} p(n)$$

where $$p(n) = \begin{cases} 1 & \text{if } n \text{ is prime} \\ 0 & \text{otherwise} \end{cases}$$

$$= \sum_{n=1}^{\infty} p(n)\, \delta\left(\frac{x}{n}\right)$$

where $$\delta(m) = \begin{cases} 0 & \text{if } 0 \le m < 1 \\ 1 & \text{otherwise} \end{cases}$$

$$= \sum_{n=1}^{\infty} p(n) \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^z}{n^z \, z} \, dz \quad , c > 0 , \quad x = \frac{1}{2} + i n \log n$$

$$= \sum_{n=1}^{\infty} p(n) \left[\frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{x^z}{n^z z} \, dz + O\left(\frac{(x/n)^c}{R \log(x/n)} \right) \right]$$

uniformly convergent for $c > 1$

$$= \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \left(\sum_{n=1}^{\infty} \frac{p(n)}{n^z} \right) \frac{x^z}{z} \, dz + O\left(\sum_{n=1}^{\infty} p(n) \frac{x^c}{n^c R \log x/n} \right)$$

Uniform convergense of $\displaystyle\sum_{n=1}^{\infty} \frac{p(n)}{n^z}$ for $|z|>1$

$$\left| \sum_{n=1}^{\infty} \frac{p(n)}{n^z} - \sum_{n=1}^{m-1} \frac{p(n)}{n^z} \right|$$

$$\leq \left| \sum_{n=m}^{\infty} \frac{p(n)}{n^z} \right| \leq \sum_{n=m}^{\infty} \frac{1}{n^c}$$

$$\sim \int_m^{\infty} \frac{1}{t^c}\, dt$$

$$= \left[\frac{t^{-c+1}}{-c+1} \right]_m^{\infty}$$

$$= \frac{1}{1-c} \frac{1}{m^{c-1}} \quad [\text{if } c>1]$$

$$S_m = \int_{c-iR}^{c+iR} \left(\sum_{n=1}^{m} \frac{p(n)}{n^z} \right) \frac{x^z}{z}\, dz$$

$$= \sum_{n=1}^{m} \int_{c-iR}^{c+iR} \frac{p(n)}{n^z} \frac{x^z}{z}\, dz$$

$$\left| S_\infty - S_m \right| = \left| \int_{c-iR}^{c+iR} \sum_{n=m+1}^{\infty} \frac{p(n)}{n^z} \frac{x^z}{z}\, d\cancel{z} \right|$$

$$\leq O\left(\frac{1}{m^{c-1}} \int_{c-iR}^{c+iR} \frac{x^z}{z}\, dz \right)$$

$$= O\left(1/m^{c-1} \right)$$

$$\Rightarrow \left| \int \left(\sum_{n=1}^{\infty} \quad \right) (\;) \, dz - \sum_{n=1}^{m} \int (\;) \, dz \right| \leq O\left(\frac{1}{m^{c-1}} \right)$$

$$\Rightarrow \lim_{m \to \infty} \left| \int \sum_{n=1}^{\infty} \quad - \sum_{n=1}^{m} \int \right| = 0$$

A cousin of $\pi(x)$

$$\psi(x) = \sum_{n=1}^{x} \Lambda(n)$$

where
$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^k \text{ for prime } p \\ 0 & \text{otherwise} \end{cases}$$

Theorem: $\psi(x) = \pi(x) \log x + O(x^{1/2})$

$$\psi(x) = \sum_{n=1}^{x} \Lambda(n)$$

$$= \sum_{n=1}^{\infty} \Lambda(n)\, \delta\left(\frac{x}{n}\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \frac{\Lambda(n)}{n^z} \frac{x^z}{z}\, dz, \quad c>1$$

$$+ O\left(\sum_{n=1}^{\infty} \frac{\Lambda(n)\, x^c}{R\, n^c \log x/n} \right)$$

$$= \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \left(\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z} \right) \frac{x^z}{z}\, dz + O(\quad)$$

Relating to ζ

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$= \prod_{\text{prime } p} \frac{1}{1 - 1/p^z}, \quad |z| > 1$$

$$\Rightarrow \log \zeta(z) = \sum_{\text{prime } p} -\log\left(1 - 1/p^z\right)$$

$$\Rightarrow \frac{\zeta'(z)}{\zeta(z)} = \sum_{\text{prime } p} - \frac{1}{1 - 1/p^z}\left(-\frac{(-\log p)}{p^z}\right)$$

$$= -\sum_{\text{prime } p} \frac{\log p}{p^z(1-1/p^z)}$$

$$= -\sum_{\text{prime } p} \frac{\log p}{p^z} \sum_{j=0}^{\infty} \frac{1}{p^{jz}}$$

$$= -\sum_{\text{prime } p} \log p \sum_{j=1}^{\infty} \frac{1}{p^{jz}}$$

$$= -\sum_{\text{prime } p} \sum_{j=1}^{\infty} \frac{\log p}{p^{jz}}$$

$$= -\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^z}$$

Therefore,

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} -\frac{\zeta'(z)}{\zeta(z)}\cdot\frac{x^z}{z}\,dz + O\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)\,x^c}{R\,n^c\log x/n}\right)$$

Error term

Error Term

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)\ x^c}{R n^c \log \frac{x}{n}} = \sum_{n=1}^{x/2} + \sum_{n=x/2}^{3x/2} + \sum_{n>3x/2}$$

$$\sum_{n=1}^{x/2} \frac{\Lambda(n)\ x^c}{R n^c \log x/n} = O\left(\sum_{n=1}^{x/2} \frac{\Lambda(n)\, x^c}{R n^c} \right)$$

$$= O\left(\frac{x^c}{R} \log x \sum_{n=1}^{x/2} \frac{1}{n^c} \right)$$

$$= O\left(\frac{x^c \log x}{R} \right)$$

$$\sum_{n > \frac{3}{2}x} \frac{\Lambda(n)\, x^c}{R\, n^c \log x/n} = O\left(\frac{x^c}{R} \sum_{n > 3/2 x} \frac{\Lambda(n)}{n^c} \right)$$

$$= O\left(\frac{x^c}{R} \right) \quad (\text{for any } c > 1)$$

$$\sum_{x/2 < n \le 3/2 x} \frac{\Lambda(n)\, x^c}{R\, n^c \log \frac{x}{n}} = O\left(\frac{x^c}{R} \sum_{x/2 < n \le 3/2 x} \frac{\Lambda(n)\, x}{n^c (x-n)} \right)$$

$$= O\left(\frac{x^c}{R} \sum_{x/2 < n \le 3/2 x} \frac{\log x}{x-n} \right)$$

let $\frac{n}{x} = 1 - s$

we have: $-1/2 \le s \le 1/2$

$$\log \frac{x}{n} = \log(1-s) = s + s^2/2 + s^3/3 + \cdots$$

$$= s + s\left[s/2 + s^2/3 + \cdots \right]$$

$$\ge 1/2\, s$$

$$= O\left(\frac{x^c}{R}\log^2 x\right).$$

Hence, error $= O\left(\dfrac{x^c \log^2 x}{R}\right)$.

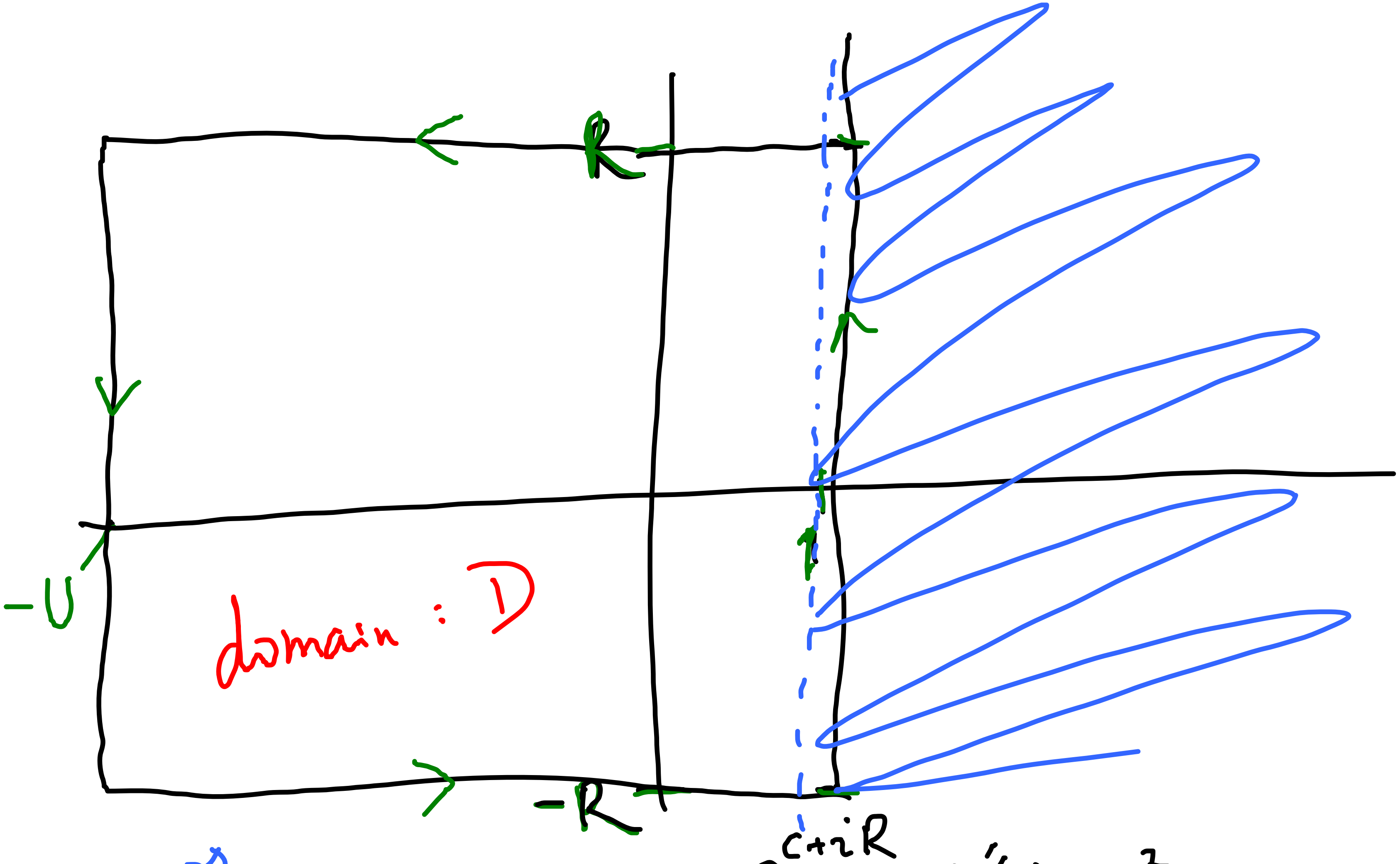
Fix $c = 1 + \dfrac{1}{\log x}$. Then:

$$\boxed{\psi(x) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} -\frac{\zeta'(z)}{\zeta(z)}\frac{x^z}{z}\,dz + O\left(\frac{x\log^2 x}{R}\right)}$$

<u>Corollary</u>: $$\psi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(z)}{\zeta(z)}\,\frac{x^z}{z}\,dz.$$

How does one evaluate

$$\int_{c-iR}^{c+iR} -\frac{\zeta'(z)}{\zeta(z)}\,\frac{x^z}{z}\,dz \quad ?$$



$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}, \quad Re(z) > 1$$

$$\int_{c-iR}^{c+iR} -\frac{\zeta'(z)}{\zeta(z)} \frac{x^z}{z} \, dz$$

<u>Assignment</u>: Prove that $\zeta(z)$ is analytic for $Re(z) > 1$.

Extending $\zeta(z)$ to $Re(z) \leq 1$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

$$\int_{y=1}^{x} \frac{\lfloor y \rfloor}{y^{z+1}} dy = \sum_{n=1}^{x-1} \int_{y=n}^{n+1} \frac{n}{y^{z+1}} dy$$

$$= \sum_{n=1}^{x-1} n\left(-\frac{1}{z}\right)\left[\frac{1}{y^z}\right]_n^{n+1}$$

$$= -\sum_{n=1}^{x-1} \frac{n}{z}\left(\frac{1}{(n+1)^z} - \frac{1}{n^z}\right)$$

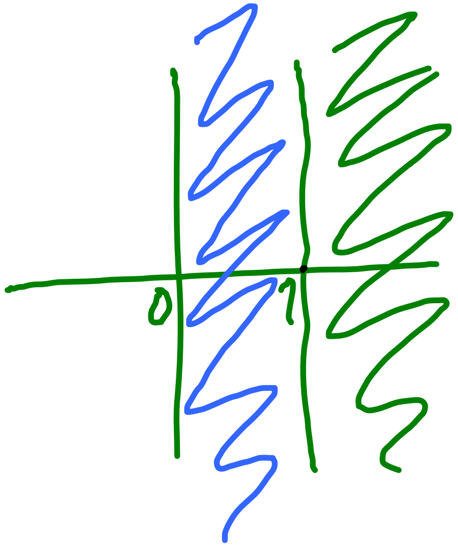
$$= -\frac{1}{z}\underbrace{\sum_{n=1}^{x-1}\left(\frac{n}{(n+1)^z} - \frac{1}{n^{z-1}}\right)}$$

$$= \frac{1}{z}\left[\sum_{n=1}^{x-1}\frac{1}{n^z} - \frac{x-1}{x^z}\right]$$

Therefore, $\displaystyle\sum_{n=1}^{x-1}\frac{1}{n^z} = \frac{x-1}{x^z} + z\int_{y=1}^{x}\frac{\lfloor y\rfloor}{y^{z+1}}\,dy$

$$\Rightarrow \boxed{\zeta(z) = z\int_{y=1}^{\infty}\frac{\lfloor y\rfloor}{y^{z+1}}\,dy}$$

$$= z\int_1^\infty \frac{y - \{y\}}{y^{z+1}}\,dy$$



$$= z\int_1^\infty \frac{dy}{y^z} - z\int_1^\infty \frac{\{y\}}{y^{z+1}}\,dy$$

$$= z\left(-\frac{1}{z-1}\,\frac{1}{y^{z-1}}\right)\Bigg|_{y=1}^{\infty} - z\int_1^\infty \frac{\{y\}}{y^{z+1}}\,dy$$

$$\Rightarrow \boxed{\zeta(z) = \frac{z}{z-1} - z\int_1^\infty \frac{\{y\}}{y^{z+1}}\,dy \quad (\text{for } \operatorname{Re}(z)>1)}$$

Analytic for $\operatorname{Re}(z) > 0$, $z \neq 1$.