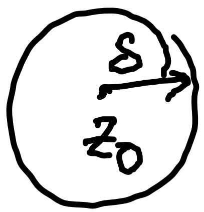
Theorem: f has an essential singularity at z_0 iff for every $w \in \mathbb{C}$, there is a sequence $\{u_n\}$ with $\lim_{n\to\infty} u_n \to z_0$ such that $\lim_{n\to\infty} f(u_n) \to w$.

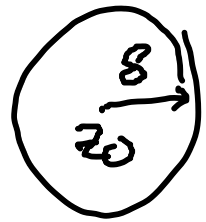
$\exists C \quad |f(z)| < C$

Removable



$\forall C \, \exists \delta \quad |f(z)| > C$

Pole



$\forall \epsilon \, \forall \delta \; \exists z$

$|z - z_0| \leq \delta$ &

$|f(z) - w| \leq \epsilon$

proof: Let $w \in \mathbb{C}$ such that $\exists \epsilon\ \exists \delta$ & $|f(z) - w| \geq \epsilon$ for every $|z - z_0| \leq \delta$.

Let $g(z) = \dfrac{1}{f(z) - w}$.

Consider $g(z)$ inside $|z - z_0| \leq \delta$.

We have $|g(z)| \leq 1/\epsilon$ in the disk.

$\Rightarrow$ g is bounded in the disk $|z - z_0| \leq \delta$.

$\Rightarrow$ g has a removable singularity at z_0.

let $g(z) = (z-z_0)^n h(z)$ with $h(z_0) \neq 0$.

$$\Rightarrow \quad f(z) = w + 1/g(z) = w + \frac{1}{(z-z_0)^n} \cdot \frac{1}{h(z)}$$

$\Rightarrow$ f has a pole of order n at z_0. ☒

Function f is <u>meromorphic</u> on domain D if on every point in D, f is either analytic or has a pole.

Integration of meromorphic functions

Suppose f has a pole at z_0 in the domain D.

Let $f(z) = \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k$, for $|z-z_0| \leq r$

Consider $\int_{|z-z_0|=r} f(z)\, dz$.

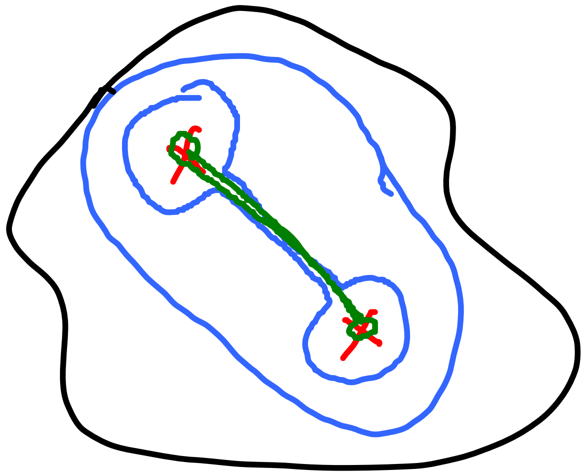
We have :

$$\int_{|z-z_0|=r} f(z)\,dz = \int_{|z-z_0|} \sum_{k=-\infty}^{\infty} a_k (z-z_0)^k \, dz$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{|z-z_0|=r} (z-z_0)^k \, dz$$

$$= a_{-1} \int_{|z-z_0|=r} \frac{dz}{z-z_0} + a_{-2} \int_{|z-z_0|=r} \frac{dz}{(z-z_0)^2} + \cdots$$

$$= 2\pi i \, a_{-1} + 0$$

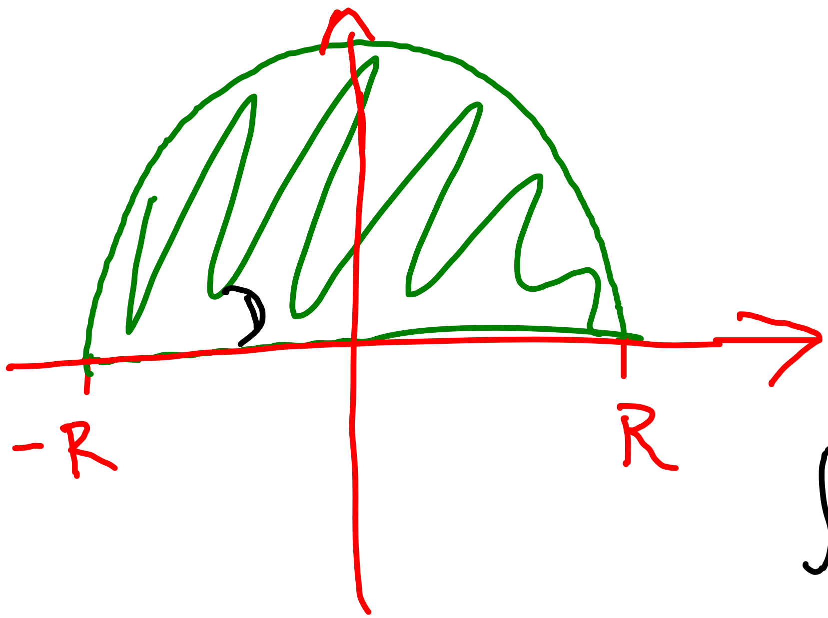


Let $\mathrm{Res}_{z_0} f(z) = a_{-1}$, where as $f(z) = \sum_{k=-\infty} a_k (z-z_0)^k$

Theorem: Let f have poles at $z_1, z_2, \dots, z_t$ inside D. Then $\int_{\delta D} f(z)\, dz = \sum_{j=1}^{t} \mathrm{Res}_{z_j}(f) \cdot 2\pi i$.

Examples

(1) $$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$



$$\int_{\delta D} \frac{dz}{1+z^2} = \int_{\delta D} \frac{dz}{(z-i)(z+i)} = \pi$$

$$\int_{\delta D} \frac{dz}{1+z^2} = \int_{-R}^{R} \frac{dx}{1+x^2} + \int_{\substack{|z|=R \\ \operatorname{Im}(z)>0}} \frac{dz}{1+z^2}$$

Computing the residues

Let $f(z) = \sum_{k=-n}^{\infty} a_k (z - z_0)^k$

$\boxed{\text{If } n = 1}$ $f(z) = a_{-1}(z - z_0)^{-1} + a_0 + \cdots$

$$\lim_{z \to z_0} f(z)(z - z_0) = a_{-1}$$

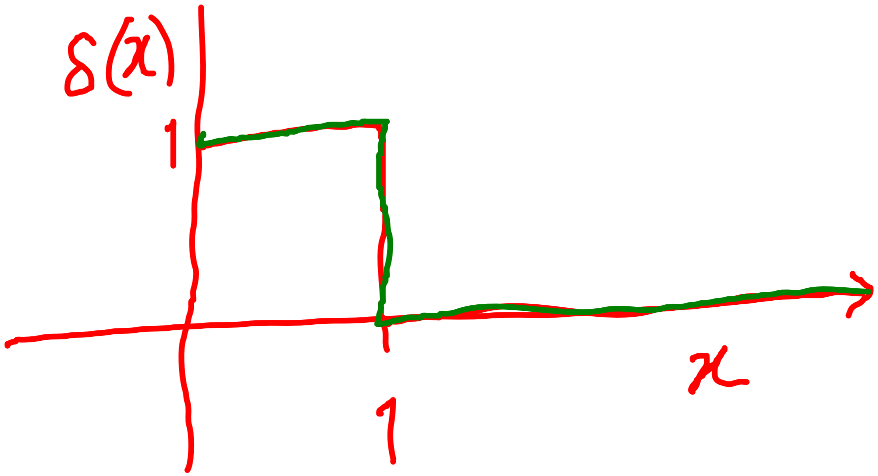
$\boxed{\text{If } n = 2}$ $\lim_{z \to z_0} f(z)(z - z_0)^2 = a_{-2}$ X

$$\lim_{z \to z_0} \left[f(z)(z - z_0)^2 \right]' = a_{-1}$$

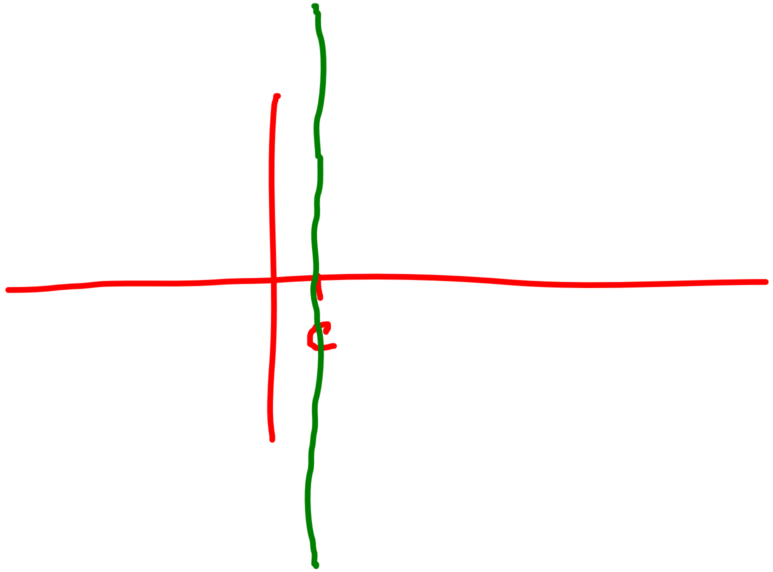
$$\left| \int_{\substack{|z|=R \\ \mathrm{Im}(z)>0}} \frac{dz}{1+z^2} \right| \leq \int_{\substack{|z|=R \\ \mathrm{Im}(z)>0}} \frac{1}{R^2-1} \, dz$$

$$\leq \frac{\pi R}{R^2-1}$$

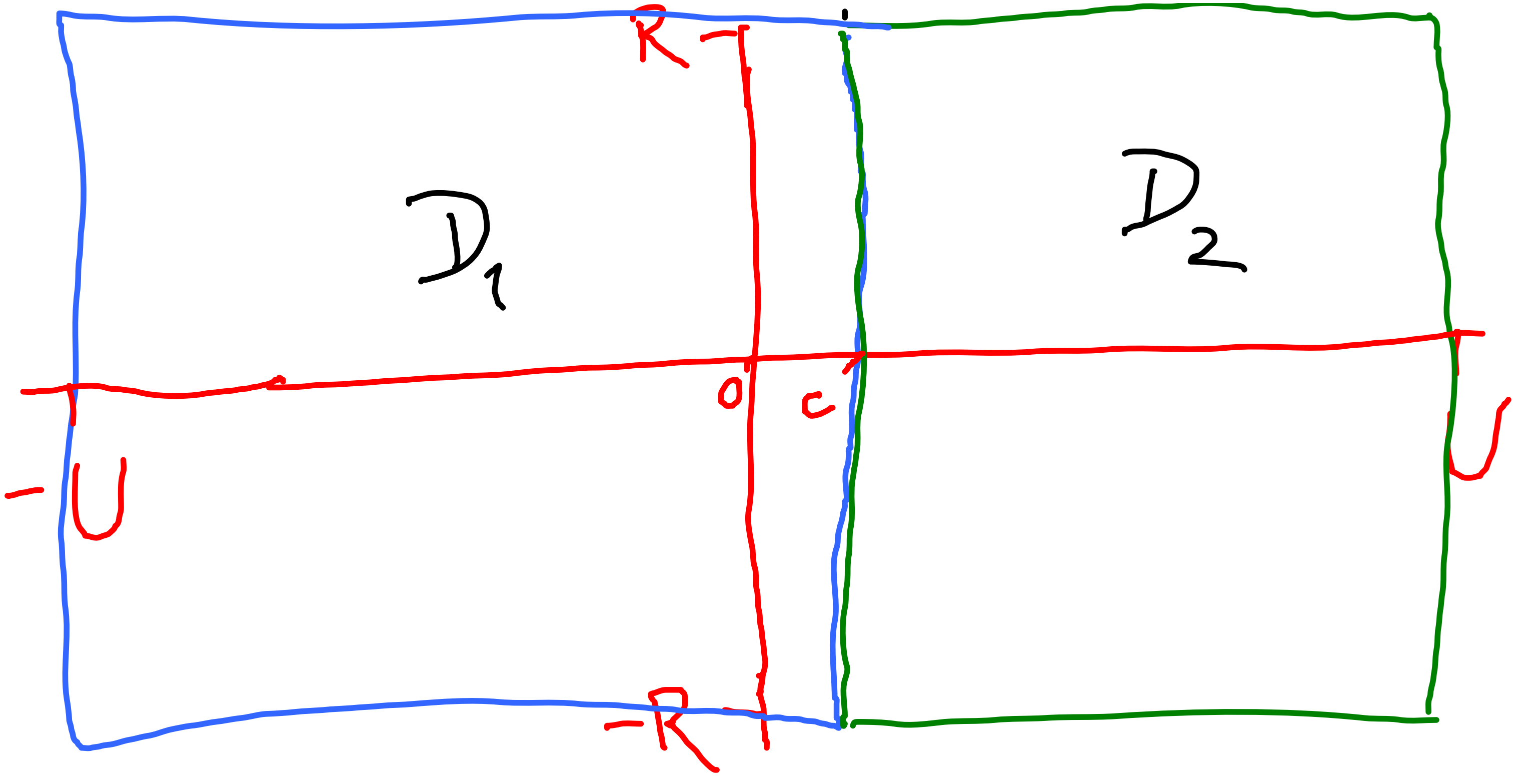
$$\Rightarrow \lim_{R\to\infty} \left| \int_{\substack{|z|=R \\ \mathrm{Im}(z)>0}} \frac{dz}{1+z^2} \right| = 0 .$$



Theorem: $$\delta(x) = \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\, ds \quad , \; c>0.$$



$$\boxed{x^s = e^{s \log x}}$$



$$\int_{\delta D_1} \frac{x^s}{s}\, ds = 2\pi i$$

$$\int_{\partial D_1} \frac{x^s}{s}\,ds = \underbrace{\int_{c-iR}^{c+iR}}_{(I_1)} + \underbrace{\int_{c+iR}^{-U+iR}}_{(I_2)} + \underbrace{\int_{-U+iR}^{-U-iR}}_{(J_3)} + \underbrace{\int_{-U-iR}^{c-iR}}_{(J_4)}$$

$$|I_2| = \left| \int_{c+iR}^{-U+iR} \frac{x^s}{s}\,ds \right| \leq \int_{c+iR}^{-U+iR} \left| \frac{x^s}{s} \right| ds$$

$$\leq \frac{1}{R} \left| \int_{c+iR}^{-U+iR} \left| e^{s \log x} \right| ds \right|$$

$$= \frac{1}{R} \left| \int_{c}^{-U} e^{t \log x}\,dt \right|$$

$$= \frac{1}{R} \left| \left[\frac{e^{t \log x}}{\log x} \right]_c^{-U} \right|$$

$$\Rightarrow \boxed{|I_2| \le \frac{1}{R} \left(\frac{x^c}{\log x} + \frac{x^{-U}}{\log x} \right)}$$

Similarly, $|I_4| \le \frac{1}{R} \left(\frac{x^c}{\log x} + \frac{x^{-U}}{\log x} \right)$

$$|I_3| \le \left| \int_{-U+iR}^{-U-iR} \frac{|e^{s \log x}|}{U} \, ds \right|$$

$$\le \left| \frac{1}{U} \int_{-U+iR}^{-U-iR} x^{-U} \, ds \right| \le \frac{2R x^{-U}}{U}$$

Therefore,

$$\int_{\delta D_1} \frac{x^s}{s}\, ds = \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds + O\left[\frac{2}{R}\left(\frac{x^c}{\log x} + \frac{x^{-v}}{\log x}\right) + \frac{2R}{v} x^{-v}\right]$$

$$\Rightarrow \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds = 2\pi i + O\left(\frac{x^c}{R \log x}\right)$$

$$\boxed{[\text{if } x > 1]}$$

$$\Rightarrow \boxed{\lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds = 2\pi i \quad [\text{if } x > 1]}$$

$$\int_{\delta D_2} \frac{x^s}{s}\, ds = 0 .$$

$$\int_{\delta D_2} \frac{x^s}{s}\, ds = \int_{c-iR}^{c+iR} + \underbrace{\int_{c+iR}^{U+iR}}_{(I_2)} + \underbrace{\int_{U+iR}^{U-iR}}_{(I_3)} + \underbrace{\int_{U-iR}^{c-iR}}_{(I_4)}$$

$$|I_4|,\ |I_2| \le \left| \int_{c+iR}^{U+iR} \frac{|e^{s \log x}|}{R}\, ds \right|$$

$$\le \left| \frac{1}{R} \int_c^U e^{t \log x}\, dt \right| \le \frac{1}{R} \left[\frac{x^U}{\log x} + \frac{x^c}{\log x} \right]$$

$$|I_3| \leq \left| \int_{U+iR}^{U-iR} \frac{|x^s|}{|s|}\, ds \right|$$

$$\leq \left| \int_{U+iR}^{U-iR} \frac{1}{U} x^U \, ds \right| \leq \frac{2R}{U} x^U$$

Therefore,

$$\int_{\delta D_2} \frac{x^s}{s}\, ds = \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds + O\left(\frac{R}{U} x^U + \frac{x^U}{R\log x} + \frac{x^c}{R \log x} \right)$$

$$= \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds + O\left(\frac{x^c}{R \log x} \right) \quad [\text{if } |x|<1]$$

$$\boxed{\Rightarrow \lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{x^s}{s}\, ds = 0 \qquad [\text{if } x<1]}$$

<u>Exercise</u>: $\displaystyle \lim_{R\to\infty} \int_{c-iR}^{c+iR} \frac{ds}{s} = i\pi$

Therefore,

$$\boxed{\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s}\, ds = \begin{cases} 1 & \text{if } x>1 \\ 1/2 & \text{if } x=1 \\ 0 & \text{if } 0<x<1 \end{cases}}$$

Example

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\sin\theta}$.

Let $z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$

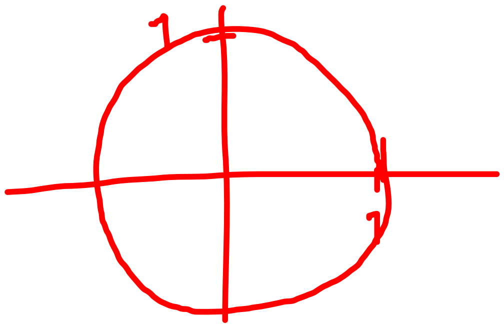
$\Rightarrow \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$

$dz = i e^{i\theta}\, d\theta = iz\, d\theta$

$$\Rightarrow \int_0^{2\pi} \frac{d\theta}{a + \sin\theta} = \int_{|z|=1} \frac{dz}{iz} \cdot \frac{1}{a + \frac{1}{2i}\left(z - \frac{1}{z}\right)}$$

complex

$$= \int_{|z|=1} \frac{dz}{aiz + \frac{1}{2} z^2 - \frac{1}{2}}$$



$$= \int_{|z|=1} \frac{2\, dz}{z^2 + 2aiz - 1}$$

$$\text{Roots of } z = \frac{-2ai \pm \sqrt{-4a^2 + 4}}{2}$$

$$= -ai \pm \sqrt{1 - a^2}$$

If $-1 < a < 1$ then

$$\int_{|z|=1} \frac{2\,dz}{z^2+2aiz-1} = 2\int_{|z|=1} \frac{dz}{(z+ai+\sqrt{1-a^2})(z+ai-\sqrt{1-a^2})}$$

$$= 2\cdot 2\pi i\left[\frac{1}{-2\sqrt{1-a^2}} + \frac{1}{2\sqrt{1-a^2}}\right]$$

$$= 0$$

If $a > 1$ then

$$\int_{|z|=1} \frac{2\,dz}{z^2+2aiz-1} = 4\pi i\left[-\frac{1}{2\sqrt{1-a^2}}\right]$$

$$= -\frac{2\pi i}{\sqrt{1-a^2}} = +\frac{2\pi}{\sqrt{a^2-1}}$$