We have:

$$\xi(z) = z(z-1)\pi^{-z/2}\Gamma(z/2)\zeta(z)$$

$$\Rightarrow \frac{\xi'(z)}{\xi(z)} = \frac{1}{z} + \frac{1}{z-1} - \frac{\log \pi}{2} + \frac{\Gamma'(z/2)}{\Gamma(z/2)} + \frac{\zeta'(z)}{\zeta(z)}$$

$$\Rightarrow \left|\frac{\zeta'(z)}{\xi(z)}\right| \leq O(\log|z|) + \left|\frac{\xi'(z)}{\xi(z)}\right|.$$

need to estimate

Order of $\xi(z)$

$$\xi(z) = 1 + z(z-1)\int_1^\infty \left(t^{z/2} + t^{(1-z)/2}\right) W(t)\,\frac{dt}{t}$$

$$\& \quad W(t) = \sum_{n\geq 1} e^{-\pi n^2 t}.$$

$$\Rightarrow \quad |\xi(z)| \leq |z|^2 \int_1^\infty \left(t^{|z|/2} + t^{1+|z|/2}\right) \frac{W(t)}{t}\,dt$$

$$= O\left(|z|^2 \int_1^\infty t^{|z|/2}\, W(t)\,dt\right)$$

$$\int_1^\infty t^{|z|/2} W(t)\,dt = \int_1^\infty t^{|z|/2} \Big(\sum_{n \geq 1} e^{-\pi n^2 t} \Big) dt$$

$$= \sum_{n \geq 1} \int_1^\infty t^{|z|/2} e^{-\pi n^2 t} dt$$

$$\leq \sum_{n \geq 1} \int_1^\infty t^{\lceil |z|/2 \rceil} e^{-\pi n^2 t} dt$$

$$\boxed{\text{Set } u = \pi n^2 t}$$

$$\leq \sum_{n \geq 1} \int_0^\infty \frac{u^{\lceil |z|/2 \rceil}}{(\pi n^2)^{\lceil |z|/2 \rceil}} \cdot e^{-u} \frac{du}{\pi n^2}$$

$$= \sum_{n \geq 1} \frac{1}{(\pi n^2)^{\lceil |z|/2 \rceil + 1}} \Gamma\left(\frac{|z|}{2}+1\right)$$

$$= O\left(\Gamma\left(\frac{|z|}{2}+1\right)\right).$$

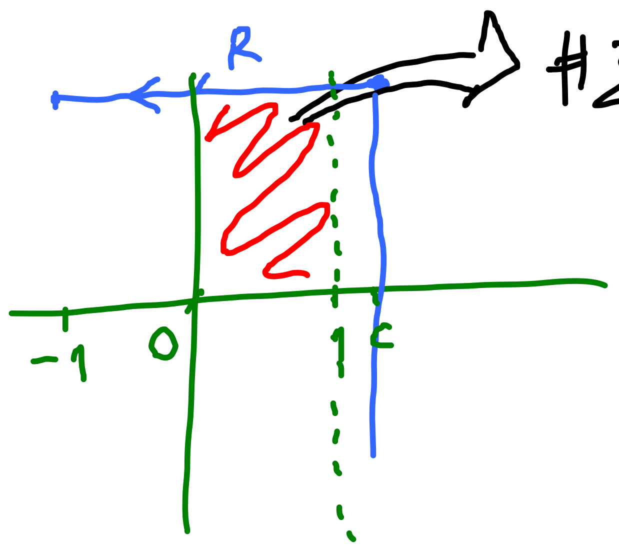
$\boxed{\text{Hence, } \xi(z) \text{ is an entire function of order } 1.}$

Using the theory of entire functions of order 1, we get:

$$\xi(z) = e^{Az+B} \prod_{\rho} \left(1 - \frac{z}{\rho}\right) e^{z/\rho}.$$

$$\Rightarrow \quad \frac{\xi'(z)}{\xi(z)} = A + \sum_{\rho} \left(\frac{-1/\rho}{1 - z/\rho} + \frac{1}{\rho} \right)$$

$$= A + \sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{z-\rho} \right)$$



$\# \text{zeros} = O(R^{1+\epsilon})$

Let $z = \alpha + iR$, $-1 \leq \alpha < 2$.

Let $\rho = \sigma + it$, $0 \leq \sigma \leq 1$.

$$\sum_\rho \left(\frac{1}{\rho} + \frac{1}{z-\rho}\right) = \sum_\rho \frac{1}{\sigma+it} + \frac{1}{(\alpha-\sigma)+i(R-t)}$$

$$= \sum_\rho \frac{\sigma - it}{|\rho|^2} + \frac{(\alpha-\sigma) - i(R-t)}{(\alpha-\sigma)^2 + (R-t)^2}$$

$$\mathrm{Re}\left[\sum_\rho \left(\frac{1}{\rho} + \frac{1}{z-\rho}\right)\right] = O(1) + \sum_\rho \frac{(\alpha-\sigma)}{(\alpha-\sigma)^2+(R-t)^2}$$

$$\Rightarrow \sum_{\rho} \frac{|\alpha-\sigma|}{(\alpha-\sigma)^2+(R-t)^2} \leq \left|\frac{\xi'(z)}{\xi(z)}\right| + O(1).$$

<u>For $z = 2+iR$:</u>

$$\left|\frac{\zeta'(z)}{\zeta(z)}\right| = O(1)$$

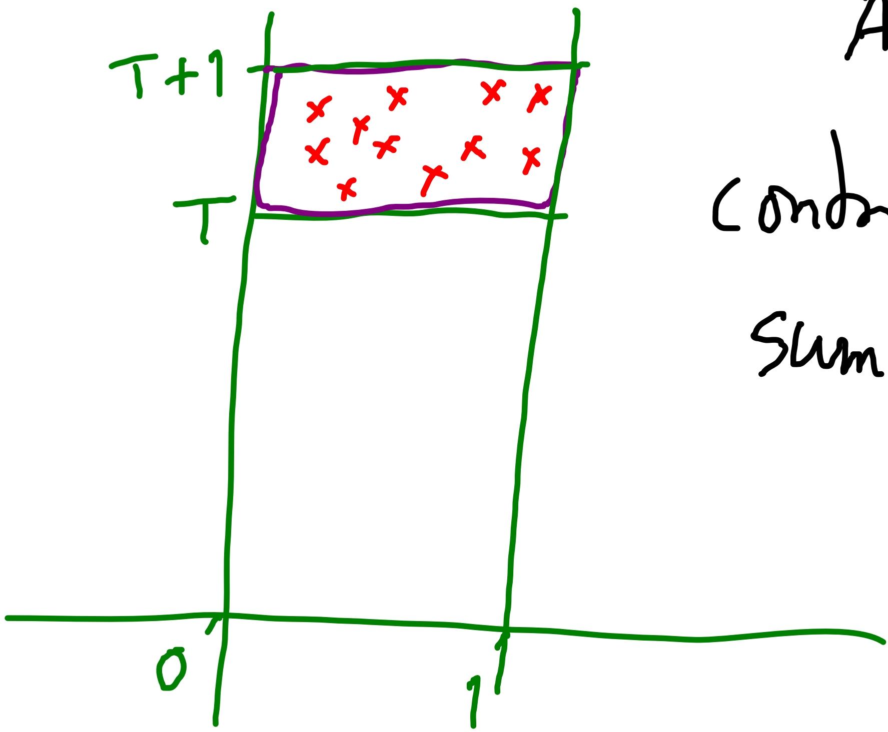
Hence, $$\left|\frac{\xi'(z)}{\xi(z)}\right| = O(\log|z|)$$
$$= O(\log R).$$

$$\Rightarrow \sum_{\rho} \frac{|2-\sigma|}{(2-\sigma)^2+(R-t)^2} = O(\log R).$$

$$\Rightarrow \sum_{\rho} \frac{1}{4+(R-t)^2} = O(\log R)$$

Therefore:

$$\boxed{\sum_\rho \frac{1}{1+(R-t)^2} = O(\log R).}$$



Any zero ρ in $[T, T+1]$ will contribute at least $1/2$ to the sum $\displaystyle\sum_\rho \frac{1}{1+(T-t)^2} = O(\log T)$.

Therefore, there are $O(\log T)$ zeros ρ in $[T, T+1]$.

This allows us to choose an R such that $z = \alpha + iR$, $-1 \leq \alpha \leq 2$, is at least distance $O\left(\frac{1}{\log R}\right)$ away from any zero ρ.

Coming back to :

$$\sum_{\rho} \left(\frac{1}{\rho} + \frac{1}{z - \rho} \right).$$

$$\triangleright \quad \frac{\zeta'(z)}{\zeta(z)} = O(\log|z|) + \sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{z-\rho}\right)$$

$$\triangleright \quad \left|\frac{\zeta'(2+iR)}{\zeta(2+iR)}\right| = O(1)$$

$$\triangleright \quad \frac{\zeta'(2+iR)}{\zeta(2+iR)} = O(\log R) + \sum_{\rho}\left(\frac{1}{\rho} + \frac{1}{2+iR-\rho}\right)$$

$$\Rightarrow \quad \frac{\zeta'(z)}{\zeta(z)} - \frac{\zeta'(2+iR)}{\zeta(2+iR)} = O(\log R) + \sum_{\rho}\left(\frac{1}{z-\rho} - \frac{1}{2+iR-\rho}\right)$$

$$\Rightarrow \left|\frac{\zeta'(z)}{\zeta(z)}\right| = O(\log R) + \left|\sum_{\rho}\left(\frac{1}{z-\rho} - \frac{1}{2+iR-\rho}\right)\right|$$

$$\left|\sum_{\rho}\left(\frac{1}{z-\rho}-\frac{1}{2+iR-\rho}\right)\right| \leq$$

$$\sum_{\rho}\frac{|z-2-iR|}{|z-\rho|\,|2+iR-\rho|} = \sum_{\rho}\frac{|\alpha-2|}{|z-\rho|\,|2+iR-\rho|}$$

$$= \sum_{\rho}\frac{|2-\alpha|}{\sqrt{\left[(\alpha-\sigma)^2+(R-t)^2\right]\left[(2-\sigma)^2+(R-t)^2\right]}}$$

$$\leq \sum_{\rho}\frac{3}{(R-t)^2}$$

$$= \sum_{\rho}\frac{6}{2(R-t)^2}$$

$$= \sum_{\substack{\rho \\ |R-t|\le 1}} \frac{3}{(R-t)^2} + \sum_{\substack{\rho \\ |R-t|>1}} \frac{6}{2(R-t)^2}$$

$$\le \sum_{\substack{\rho \\ |R-t|\le 1}} \frac{3}{(R-t)^2} + \sum_{\substack{\rho \\ |R-t|\ge 1}} \frac{6}{1+(R-t)^2}$$

$$= \sum_{\substack{\rho \\ |R-t|\le 1}} \frac{3}{(R-t)^2} + O(\log R)$$

$$\le \sum_{\substack{\rho \\ |R-t|\le 1}} O(\log^2 R) = O(\log^3 R).$$

$$\left| \int_{c+iR}^{-1+iR} \frac{\zeta'(z)}{\zeta(z)} \cdot \frac{x^z}{z}\, dz \right|$$

$$\leq \left| \int_{c+iR}^{-1+iR} O(\log^3 R)\, \frac{x^\alpha}{R}\, d\alpha \right|$$

$$= O\left(\frac{\log^3 R}{R}\right) \left| \int_{c}^{-1} x^\alpha\, d\alpha \right|$$

$$= O\left(\frac{\log^3 R}{R} \left[\frac{x^c}{\log x} + \frac{1}{x \log x} \right] \right)$$

$$= O\left(\frac{x \log^3 R}{R \log x} \right).$$

Going back all the way :

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} \frac{\zeta'(z)}{\zeta(z)} \cdot \frac{x^z}{z}\, dz + O\left(\frac{x \log^2 x}{R}\right)$$

We now know that :

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR} \frac{\zeta'(z)}{\zeta(z)} \cdot \frac{x^z}{z}\, dz = x + \sum_{m\geq 1} \frac{x^{-2m}}{2m} - \sum_{\substack{\rho \\ -R\leq t\leq R}} \frac{x^\rho}{\rho} + O\left(\frac{x \log^3 R}{R \log x} + \frac{R^\epsilon}{xR\log x}\right)$$

$$\sum_{m\geq 1} \frac{x^{-2m}}{2m} = + \frac{1}{2}\sum_{m\geq 1} \frac{x^{-2m}}{m} = -\frac{1}{2}\log\left(1-\frac{1}{x^2}\right)$$

Therefore:

$$\psi(x) = x - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) - \sum_{\substack{\rho \\ -R\le t\le R}} \frac{x^\rho}{\rho} + O\left(\frac{x\log^3 R}{R\log x} + \frac{R^\epsilon}{xR\log x} + \frac{x\log^2 x}{R}\right)$$

$$\Rightarrow \boxed{\psi(x) = x - \frac{1}{2}\log\left(1-\frac{1}{x^2}\right) - \sum_{\rho} \frac{x^\rho}{\rho}}$$

Fixing $R = x^{1/2}$:

$$\psi(x) = x - \sum_{\substack{\rho \\ -R \le t \le R}} \frac{x^\rho}{\rho} + O(x \ln^2 x)$$

$\boxed{\text{Riemann Hypothesis}}$: For all ρ, $\mathrm{Re}(\rho) = \frac{1}{2}$.

(Assuming RH)

$$\left| \sum_{\substack{\rho \\ -R \le t \le R}} \frac{x^\rho}{\rho} \right| \le \sum_{\substack{\rho \\ -R \le t \le R}} \frac{x^{1/2}}{|\rho|}$$

$$= O\left(x^{1/2} \sum_{\substack{\rho \\ 1 \le t \le R}} \frac{1}{|\rho|} \right)$$

$$= O\left(x^{1/2} \sum_{1 \le t \le R} \frac{\log t}{t}\right)$$

$$= O\left(x^{1/2} \log^2 R\right)$$

Therefore, if RH is true, then

$$\psi(x) = x + O\left(x^{1/2} \log^2 x\right).$$

__Theorem:__ If $\psi(x) = x + O(x^{1/2+\epsilon})$ for any $\epsilon > 0$, then Riemann Hypothesis is true.

proof: We know that:

$$\zeta(z) = \prod_{\text{prime } p} \left(\frac{1}{1 - \frac{1}{p^z}} \right), \quad \text{for } \operatorname{Re}(z) > 1.$$

$$\Rightarrow \frac{\zeta'(z)}{\zeta(z)} = - \sum_{\text{prime } p} \frac{\log p}{1 - \frac{1}{p^z}}$$

$$= -\sum_{\text{prime } p} \log p \sum_{k \geq 0} \frac{1}{p^{kz}}$$

$$= -\sum_{n>0} \frac{\Lambda(n)}{n^z},$$

where $\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m, p \text{ prime} \\ 0 & \text{otherwise} \end{cases}$

Also, $\psi(x) = \sum_{n \leq x} \Lambda(n)$.

$$\Rightarrow \Lambda(n) = \psi(n) - \psi(n-1)$$

Therefore,

$$\frac{\zeta'(z)}{\zeta(z)} = -\sum_{n>0} \frac{\psi(n)-\psi(n-1)}{n^z}$$

$$= -\sum_{n>0} \psi(n)\left[\frac{1}{n^z} - \frac{1}{(n+1)^z}\right]$$

$$= -\sum_{n>0} \int_n^{n+1} z\,\frac{\psi(t)}{t^{z+1}}\,dt$$

$$= -z\int_1^{\infty} \frac{\psi(t)}{t^{z+1}}\,dt, \quad Re(z)>1.$$

Since $\psi(t) = t + O\left(t^{1/2+\epsilon}\right)$,

$$\frac{\zeta'(z)}{\zeta(z)} = -z \int_1^\infty \frac{t + O\left(t^{1/2+\epsilon}\right)}{t^{z+1}}\, dt$$

$$= -z \int_1^\infty \frac{dt}{t^z} - O\left[z \int_1^\infty \frac{dt}{t^{z+1/2-\epsilon}}\right]$$

$$= -z \left[\frac{t^{-z+1}}{-z+1}\right]_1^\infty - O\left(z \int_1^\infty \frac{dt}{t^{z+1/2-\epsilon}}\right)$$

$$= -\frac{z}{z-1} - O\left(z \int_1^\infty \frac{dt}{t^{z+1/2-\epsilon}}\right)$$

RHS is analytic for $\mathrm{Re}(z) > \frac{1}{2} + \epsilon$,
except for pole at $z = 1$.

$\Rightarrow$ $\dfrac{\zeta'(z)}{\zeta(z)}$ is analytic for $\mathrm{Re}(z) > \frac{1}{2} + \epsilon$
except for pole at $z = 1$.

$\Rightarrow$ RH is true. $\boxtimes$

Relationship between $\psi(x)$ & $\pi(x)$

$$\pi(x) + \frac{1}{2}\pi(x^{1/2}) + \frac{1}{3}\pi(x^{1/3}) + \cdots .$$

$$= \pi(x) + O(x^{1/2}) = \int_1^x \frac{d\psi(t)}{\lg t} \longrightarrow = \frac{x}{\lg x} + O\left(\frac{x^{1/2} S(x)}{\lg x}\right)$$

$$d\psi(t) = \begin{cases} 0 & \text{mostly} \\ \Lambda(t) & \text{at } t = \text{integer} \end{cases}$$

Let $\psi(t) = t + O\left(t^{1/2}\delta(t)\right)$.

Then,

$$\int_1^x \frac{d\psi(t)}{\log t} = \int_1^x dt\,\frac{1 + O\left[t^{-1/2}\delta(t) + t^{1/2}\delta'(t)\right]}{\log t}$$

$$= \int_1^x \frac{dt}{\log t} + O\left[\int_1^x \frac{t^{-1/2}\delta(t) + t^{1/2}\delta'(t)}{\log t}\,dt\right]$$

Assume that $\delta(t) = \log^2 t$.

Then,

$$\int_1^x \frac{t^{-1/2}\delta(t) + t^{1/2}\delta'(t)}{\log t}\, dt = \int_1^x \left(t^{-1/2}\log t + 2t^{-1/2}\right) dt$$

$$\int_1^x \frac{\log t}{\sqrt{t}}\, dt = \left[2\sqrt{t}\log t\right]_1^x + \int_1^x \frac{2}{\sqrt{t}}\, dt$$

$$= O\left(x^{1/2}\log x\right).$$