In $\psi(x)$, prime p contributes $\lfloor \log_p x \rfloor \cdot \log p$.

Approach not likely to work.

$$\lfloor \log_p x \rfloor \cdot \log p = \left\lfloor \frac{\log x}{\log p} \right\rfloor \cdot \log p$$

$$= \left[\frac{\log x}{\log p} + O(1) \right] \log p$$

$$= \log x + O(\log p)$$

$$\Rightarrow \psi(x) = \sum_{\text{prime } p \le x} \left[\log x + O(\log p) \right]$$

$$= \pi(x) \log x + O\left(\log \prod_{\text{prime } p \le x} p \right)$$

$$\psi(x) = \frac{1}{2\pi i}\int_{c-iR}^{c+iR} -\frac{\zeta'(z)}{\zeta(z)}\cdot\frac{x^z}{z}\,dz + \cancel{O\left(\frac{x\log^2 x}{R}\right)}$$

$$\sum_{n>0}\frac{\Lambda(n)\,x^c}{R\,n^c\log\frac{x}{n}}$$

$$+\frac{1}{2\pi i}\left[\int_{c+iR}^{\infty+iR} + \int_{c-iR}^{\infty-iR}\frac{x^z}{z}\,dz\right]$$

$$-\frac{\zeta'}{\zeta} = \sum_{n\geq 1} \frac{\Lambda(n)}{n^z}$$

$$\psi(x) = \sum_{n\leq x} \Lambda(n)$$

$$= \sum_{n\geq 1} \Lambda(n)\,\delta\left(\frac{x}{n}\right)$$

$$= \sum_{n\geq 1} \Lambda(n) \left[\frac{1}{2\pi i}\int_{c-iR}^{c+iR} + \int_{c+iR}^{\alpha+iR} + \int_{\alpha-iR}^{c-iR} \frac{x^z}{n^z z}\,dt\right]$$

$$= \frac{1}{2\pi i}\int_{c-iR}^{c+iR} -\frac{\zeta'}{\zeta}\frac{x^z}{z}\,dz + \frac{1}{2\pi i}\left[\int_{c+iR}^{\alpha+iR} + \int_{\alpha-iR}^{c-iR} \sum_{n\geq 1}\frac{\Lambda(n)}{n^z}\frac{x^z}{z}\,dt\right]$$

$$= x - \frac{\zeta'(0)}{\zeta(0)} - \ln\left(1-\frac{1}{x^2}\right) + \sum_{\substack{\rho \\ |\operatorname{Im}(\rho)| \leq R}} \frac{x^\rho}{\rho}$$

$$+ \frac{1}{2\pi i}\left[\int_{c+iR}^{\infty+iR} + \int_{\infty-iR}^{c-iR}\right] \sum_{n\geq 1} \frac{\Lambda(n)}{n^z} \frac{x^z}{z}\, dz$$

$$\Rightarrow \frac{d\psi(x)}{dx} = 1 - \frac{2/x^3}{1-1/x^2} + \sum_{\substack{\rho \\ |\operatorname{Im}(\rho)| \leq R}} x^{\rho-1}$$

$$+ \frac{1}{2\pi i}\left[\int + \int\right]\left(\sum_{n\geq 1}\frac{\Lambda(n)}{n^z}\right) x^{z-1}\, dz$$

$$= 1 + O\left(\frac{1}{x^3}\right) + O\left(\frac{R\ln R}{x^{1/2}}\right) + O\left(\int_c^\infty \sum_{n\geq 1} \frac{\Lambda(n)}{n^t} x^{t-1}\, dt\right)$$

$$\int_c^{\delta} \sum_{n \geq 1} \Lambda(n) \frac{x^t}{n^t}\, dt = \sum_{n \geq 1} \frac{\Lambda(n)}{x} \int_c^{\delta} \left(\frac{x}{n}\right)^t dt$$

Does not work!

$$= \sum_{n \geq 1} \frac{\Lambda(n)}{x} \left(\frac{x}{n}\right)^c \frac{1}{\ln \frac{x}{n}}$$

$$= O\left(\ln^2 x\right)$$

Therefore, $$\frac{d\Psi(x)}{dx} = 1 + O\left(\frac{R \ln R}{x^{1/2}}\right) + O\left(\ln^2 x\right)$$

For $R = x^{1/2}$, $$\frac{d\Psi(x)}{dx} = 1 + O\left(\ln^2 x\right)$$

We have :

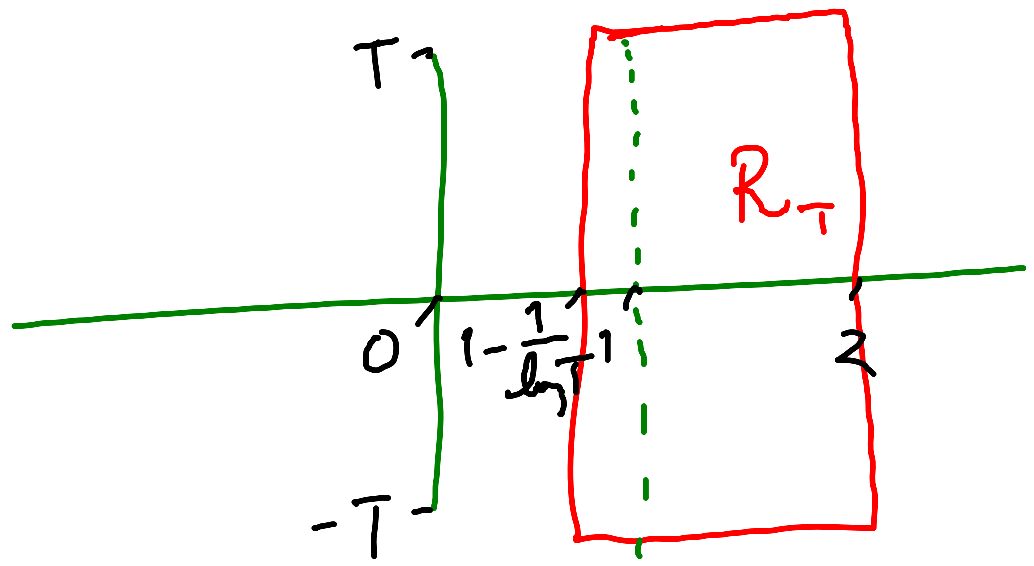
$$\int_1^x \frac{d\psi(t)}{\log t} = \pi(x) + O\left(\pi\left(x^{1/2}\right)\right)$$

$$\int_1^x \frac{d\psi(t)}{\log t} = \int_1^x \frac{1}{\log t}\, \psi'(t)\, dt$$

$$= \int_1^x \frac{dt}{\log t} + O\left(\int_1^x \log t \, dt\right)$$

Assignment : Fix this!

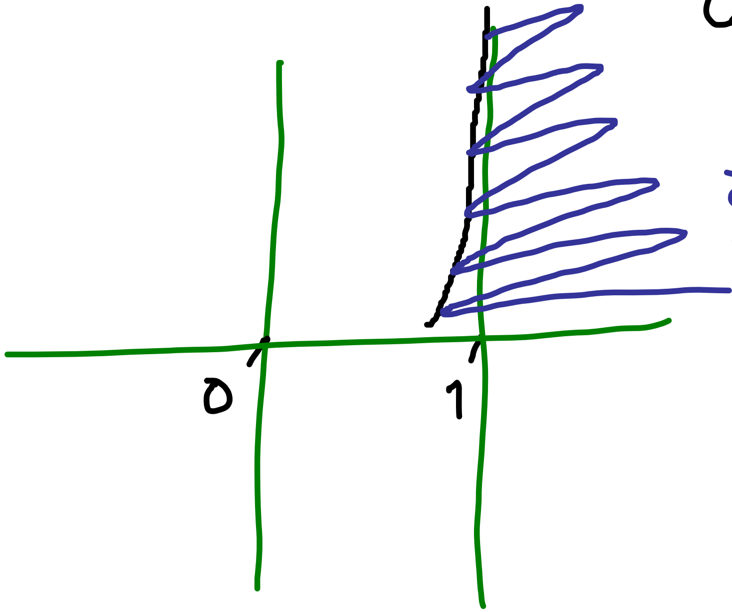
Prime Number Theorem



Fact: (1) On δR_T, $|\zeta(z)| = O(\log T)$

(2) On δR_T, $|\zeta'(z)| = O(\log^2 T)$

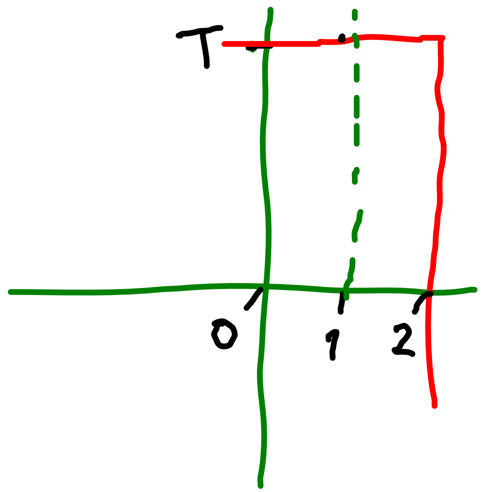
Theorem: $\zeta(z) \neq 0$ for $z = \sigma + iT$, $\sigma \geq 1 - O\left(\frac{1}{\log^7 T}\right)$.



$\zeta(z) \neq 0$.

Corollary: $\psi(x) = x + O\left(x^{1-o(1)}\right)$.

proof: $[z=\sigma+iT]$ $\zeta(z) = \prod_{\text{prime } p} \left(\frac{1}{1-\frac{1}{p^z}} \right)$



for $1 < \sigma \leq 2$.

$$\Rightarrow \log \zeta(z) = -\sum_{\text{prime } p} \log\left(1-\frac{1}{p^z}\right)$$

$$= \sum_{\text{prime } p} \sum_{m \geq 1} \frac{1}{m p^{mz}}$$

$$= \sum_{n > 1} \frac{\Lambda(n)}{\log n \, n^z}.$$

$$\Rightarrow \quad \underset{\displaystyle \log|\zeta(z)|}{\underset{\shortparallel}{\mathrm{Re}\left[\log \zeta(z)\right]}} = \sum_{n>1} \frac{\Lambda(n)}{\log n \, n^{\sigma}} \cos(T \log n)$$

<u>We know</u>: $\cos 2\theta = 2\cos^2\theta - 1$

$$= 2\cos^2\theta + 4\cos\theta + 2 - 4\cos\theta - 3$$

$$\Rightarrow \quad 3 + 4\cos\theta + \cos 2\theta = 2(\cos\theta + 1)^2 \geq 0.$$

$$\Rightarrow \quad 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+iT)| + \log|\zeta(\sigma+2iT)| \geq 0.$$

$$\Rightarrow \log\left[|\zeta(\sigma)|^3 |\zeta(\sigma+iT)|^4 |\zeta(\sigma+2iT)|\right] \geq 0$$

$$\Rightarrow \boxed{|\zeta(\sigma)|^3 |\zeta(\sigma+iT)|^4 |\zeta(\sigma+2iT)| \geq 1.}$$

As $\sigma \to 1$, $\zeta(\sigma) \to \infty$.

If $\zeta(1+iT)=0$, then the above product $\to 0$ as $\sigma \to 1$.

Therefore, $\zeta(1+iT) \neq 0$.

Also :

$$|\zeta(\sigma+iT)|^4 \geq \frac{1}{|\zeta(\sigma)|^3 |\zeta(\sigma+2iT)|}$$

for $1 < \sigma \leq 2$.

Assume $\sigma = 1 + \frac{c_1}{(\log T)^9}$.

Near $\alpha = 1$, $|\zeta(\alpha)| \sim \frac{1}{\alpha - 1}$

$$\Rightarrow |\zeta(\sigma)| = O(\log^9 T)$$

Also, $|\zeta(\sigma+2iT)| = O(\log T)$.

Therefore,

$$|\zeta(\sigma+iT)|^4 \geq O(\log^{-28} T)$$

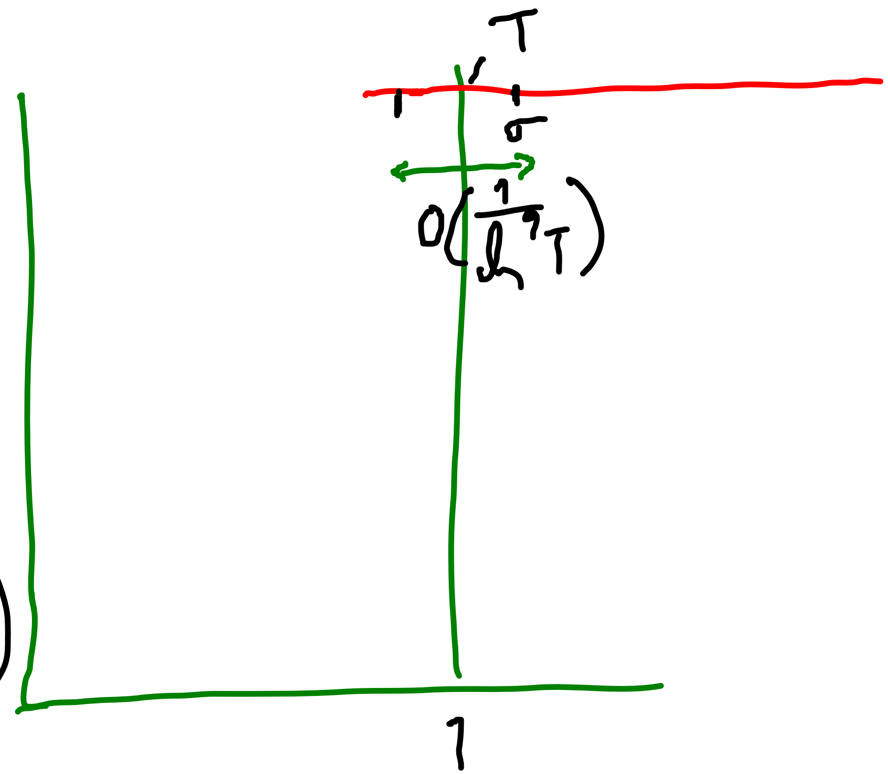
$$\Rightarrow |\zeta(\sigma+iT)| \geq O\left(\frac{1}{\log^7 T}\right).$$

We know that

$$\zeta'(\alpha + iT) = O(\ln^2 T)$$

for $\sigma \geq \alpha \geq 2-\sigma$,

and $\zeta(\sigma + iT) \geq O\left(\frac{1}{\ln^7 T}\right)$



Choose c_1 such that $\zeta(2-\sigma+iT) > 0$.

This is always possible since the reduction will be at most $\frac{c_1 c'}{\ln^7 T}$. $\square$