Dirichlet's Theorem

Theorem: Let $a, q \in \mathbb{N}$ and $(a, q) = 1$.

Let $$\psi(x, a, q) = \sum_{\substack{\text{prime } p \leq x \\ p \equiv a\ (q)}} \Lambda(p).$$

Then, $$\psi(x, a, q) = \frac{1}{\phi(q)} x \left(1 + o(1)\right).$$

Charecters mod q

$\chi : \mathbb{Z}_q^* \longmapsto \mathbb{C}$, χ is a charecter

is it is multiplicative.

<u>Facts</u>: (1) $\chi(a)$ is a $\phi(q)^{th}$ root of unity.

(2) Let $G = \{ \chi \mid \chi \text{ a character} \}$.

Then G is a group

(3) $|G| = \phi(q)$.

(4) $$\sum_{a \in \mathbb{Z}_q^*} \chi(a) = \begin{cases} \phi(q) & \text{if } \chi = \chi_0 \\ 0 & \text{otherwise} \end{cases}$$

where $\chi_0(a) = 1$.

(5) $$\sum_{\chi \in G} \chi(a) = \begin{cases} \phi(q) & \text{if } a = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$L(z, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^z}$$

where $\chi(n) = \begin{cases} \chi(a) & \text{for } a = n(q) \ \& \ (n,q) = 1 \\ 0 & \text{for } (n,q) > 1. \end{cases}$

$$L(z, \chi_0) = \sum_{\substack{n \geq 1 \\ (n,q)=1}} \frac{1}{n^z}$$

$$= \prod_{\substack{\text{prime } p \\ (p,q)=1}} \frac{1}{1 - \frac{1}{p^z}}$$

Similarly,

$$L(z, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^z}$$

$$= \prod_{\substack{\text{prime } p \\ (p,q)=1}} \frac{1}{1 - \frac{\chi(p)}{p^z}}$$

$$L(1, \chi_0) = \sum_{\substack{n \geq 1 \\ (n,q)=1}} \frac{1}{n}$$

Let $p_1, p_2, \ldots, p_\ell \mid q$, and no other prime.

$$\sum_{\substack{n \geq 1 \\ (n,q)=1}} \frac{1}{n} = \zeta(1) - \sum_{i=1}^{\ell} \frac{1}{p_i} \zeta(1)$$

$$+ \sum_{\substack{i,j=1 \\ i \neq j}}^{\ell} \frac{1}{p_i p_j} \zeta(1)$$

$$- \ldots$$

$$= \zeta(1)\left[1 - \sum_{i} \frac{1}{p_i} + \sum_{\substack{i,j \\ i \neq j}} \frac{1}{p_i p_j} - \cdots\right]$$

$$= \zeta(1) \prod_{i=1}^{\ell} \left(1 - \frac{1}{p_i}\right).$$

Hence, $L(1, \chi_0)$ diverges.

What about $L(1, \chi)$, $\chi \neq \chi_0$?

$$L(1, \chi) = \sum_{\substack{n \geq 1 \\ (n,q)=1}} \frac{\chi(n)}{n}$$

$$\left| \sum_{\substack{N \geq n \geq 1 \\ (n,q)=1}} \chi(n) \right| \leq \phi(q), \text{ for any } N > 0.$$

Consider $\displaystyle\sum_{\substack{Nq < n \leq (N+1)q \\ (n,q)=1}} \frac{\chi(n)}{n}$

$$= \sum_{\substack{0 < n \leq q \\ (n,q)=1}} \frac{\chi(n)}{Nq+n}$$

$$= \frac{1}{Nq} \left[\sum_{\substack{0 < n \leq q \\ (n,q)=1}} \frac{\chi(n)}{1 + \frac{n}{Nq}} \right].$$

Assignment

prove that $L(1,\chi) \downarrow$ formally.

Functional equation

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

$$t \to \pi n^2 u$$

$$\Gamma\left(\frac{z}{2}\right) = \int_0^\infty (\pi n^2 u)^{\frac{z}{2}-1} e^{-\pi n^2 u}\, \pi n^2 du$$

$$= \pi^{z/2} n^z \int_0^\infty u^{z/2-1} e^{-\pi n^2 u}\, du$$

$$\Gamma(z/2)\,\pi^{-z/2}\,\frac{1}{n^z} = \int_0^\infty u^{z/2-1} e^{-\pi n^2 u}\,du$$

$$\Rightarrow \Gamma(z/2)\,\pi^{-z/2}\,L(z,\chi) = \int_0^\infty u^{z/2-1} \sum_{n\geq 1} \chi(n) e^{-\pi n^2 u}\,du$$

<u>Odd charater</u> : $\chi(-1) = -1$

<u>Even character</u> : $\chi(-1) = 1$

<u>Theorem</u>: If χ is an even character, then, letting,

$$\xi(z,\chi) = \pi^{-z/2}\,\Gamma(z/2)\,L(z,\chi)$$

$$\boxed{\xi(z,\chi) = \frac{\tau(\chi)}{\sqrt{q}}\,\xi(1-z,\overline{\chi})}$$

where $\tau(\chi) = \sum_{m=1}^{q} \chi(m)\, e^{2\pi i \frac{m}{q}}$.

Theorem: If χ is an odd character, then, letting,

$$\xi(z,\chi) = \pi^{-z/2-1}\,\Gamma\left(\tfrac{z}{2}+1\right) L(z,\chi)$$

$$\boxed{\xi(z,\chi) = \frac{\tau(\chi)}{i\sqrt{q}}\,\xi(1-z,\overline{\chi})}$$

let $\psi(x, \chi) = \sum_{n \le x} \chi(n) \Lambda(n)$.

$$\psi(x, \chi) = \sum_{n \ge 1} \chi(n) \Lambda(n) \, \delta\left(\frac{x}{n}\right)$$

$$\vdots$$

$$= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} -\frac{L'(z, \chi)}{L(z, \chi)} \frac{x^z}{z} \, dz$$

$$= \frac{1}{2\pi i} \int_{c-iR}^{c+iR} -\frac{L'}{L}(z, \chi) \frac{x^z}{z} \, dz + \text{error}$$

Eventually:

$$\psi(x, \chi_0) = x + \text{error}$$

$$\& \quad \psi(x, \chi) = \text{error}$$
$$\chi \neq \chi_0$$

Generalized Riemann Hypothesis

All non-trivial zeros of $L(z,\chi)$ lie on the line $\mathrm{Re}(z) = 1/2$.

Assuming GRH:

$$\psi(x,\chi_0) = x + O\left(x^{1/2}\log^2 qx\right)$$

$$\& \quad \psi(x,\chi) = O\left(x^{1/2}\log^2 qx\right)$$
$$\chi \neq \chi_0$$

We are interested in:

$$\psi(x, a, q) = \sum_{\substack{n \leq x \\ n = a\ (q)}} \Lambda(n)$$

Consider $\bar{\chi}(a)\psi(x, \chi)$

$$= \bar{\chi}(a) \sum_{n \leq x} \chi(n)\Lambda(n)$$

$$= \sum_{n \leq x} \chi\left(\frac{n}{a}\right)\Lambda(n)$$

So,
$$\sum_{\chi \in G} \bar{\chi}(a)\, \psi(x, \chi)$$
$$= \sum_{\chi \in G} \sum_{n \le x} \chi\left(\frac{n}{a}\right) \Lambda(n)$$
$$= \sum_{n \le x} \Lambda(n) \sum_{\chi \in G} \chi\left(\frac{n}{a}\right)$$
$$= \phi(q) \sum_{\substack{n \le x \\ n = a (q)}} \Lambda(n) = \phi(q)\, \psi(x, a, q).$$

Therefore,

$$\psi(x, a, q) = \frac{1}{\phi(q)} \sum_{\chi \in G_q} \bar{\chi}(a)\, \psi(x, \chi)$$

$$= \frac{1}{\phi(q)} \psi(x, \chi_0) + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \bar{\chi}(a)\, \psi(x, \chi)$$

$$= \frac{x}{\phi(q)} + O\left(x^{1/2} \log^2 qx\right),$$

$\boxed{\text{assuming GRH}}$

Theorem: Let H be a proper subgroup of $\mathbb{Z}_q^*$. Then there is an element in $\mathbb{Z}_q^* \setminus H$ in the range $[1, \log^4 q]$.

proof: Let χ be a non-trivial character such that $\chi(H) = 1$.

Consider $$\sum_{n \leq x} (1 - \chi(n)) \Lambda(n).$$

Suppose $[1, x] \cap \mathbb{Z}_q^{\times} \subseteq H$.

Then $$\sum_{\substack{n \le x \\ x < q}} (1 - \chi(n)) \Lambda(n) = 0.$$

So, if $$\sum_{x \le \log^4 q} (1 - \chi(n)) \Lambda(n) \neq 0$$

then we are done.

$$\sum_{n \le x} (1 - \chi(n)) \Lambda(n)$$

$$= \sum_{n \le x} \Lambda(n) - \sum_{n \le x} \chi(n) \Lambda(n)$$

$$= \psi(x) - \psi(x, \chi)$$

$$= x + O(x^{1/2} \log^2 x) + O(x^{1/2} \log^2 qx)$$

$$= x + O(x^{1/2} \log^2 qx)$$

$$x > c\, x^{1/2} \log^2 qx$$

$$\Updownarrow$$

$$x > c^2 \log^4 qx$$

So, if $x > 10c^2 \log^4 q$ then

$$x > c^2 \log^4 q + c^2 \log x. \qquad \boxtimes$$

<u>Corollary</u>: Numbers $\leq O(\log^4 q)$ generate $\mathbb{Z}_q^*$.

Has many applications:

(1) Finding a quadratic non-residue in $\mathbb{F}_p$

(2) Miller's algorithm for primality testing