CS746: Endsem Examination

Maximum Marks: 40

Due on Nov 27, 2012

NOTE: Feel free to discuss the questions and answers with each other. However, everyone needs to write the answers separately. Also, please submit answers in a PDF file, preferably not scans of handwritten text!

Question 1 (marks 5+15). Let f be analytic inside domain C. Prove that the number of zeros of f inside C equals $\frac{1}{2\pi}\Delta_C \arg(f)$ where $\Delta_C \arg(f)$ is the total change in argument of $f(z)$ when moving along C.

Let $N(T)$ be the number of zeros of $\zeta(z)$ in the rectangle $0 < \Re(z) < 1$ and $0 < \Im(z) < T$. Prove that

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

Hint: Let $f = \xi$, C be the rectangle with vertices 2, $2 + iT$, $-1 + iT$, -1 and use above.

Question 2 (marks 10). Let χ be a *primitive* character modulo q (primitive means $\chi^k \neq 1$ for any $k < \phi(q)$; all the results we discussed in class regarding L-functions hold only for primitive characters as non-primitive characters are characters also of some $p < q$, $p|q$). Assume that $\chi(-1) = 1$. Let

$$\xi(z, \chi) = (q/\pi)^{z/2}\Gamma(z/2)L(z, \chi).$$

Prove that $L(z, \chi)$ has infinitely many zeros in the strip $0 \leq \Re(z) \leq 1$ and

$$\xi(z, \chi) = e^{Az+B}\prod_{\rho}(1 - \frac{z}{\rho}) \cdot e^{z/\rho},$$

where ρ ranges over non-trivial zeros of $L(z, \chi)$. You may assume that $\xi(z) = \xi(1 - z)$.

Question 3 (marks 10). Define

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz + n)^k},$$

for $k \geq 3$. Prove that

$$G_k(\tau z) = (cz + d)^k G_k(z),$$

for $\tau \in SL_2(Z)$, $\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.