Equivalence between two definitions of determinant

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1 Introduction

There are more than one definitions of the determinant of a matrix. In this document, we will reduce one definition to another. Though I knew both the definitions for a long time, this time I could not recall why the definitions were the same. Hence, after I got the (pretty simple) answer I decided to TeX it.

A determinant for an $n \times n$ matrix $A = \left[a_{ij}\right], 1 \leq i, j \leq n$ over some commutative ring[1] is defined as follows:

Definition 1 (Permutation based definition of determinant).

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$

where S_n is the symmetric group of order n.

A recursive definition of the determinant is given by:

Definition 2 (Minor based definition of determinant).

$$\det(A) = \begin{cases} \sum_{i=1}^{n} (-1)^{i-1} a_{1i} \cdot \det(A_{1i}) & \textit{if } n > 1, \\ a_{11} & \textit{if } n = 1, \end{cases}$$

where A_{1i} is the $(n-1) \times (n-1)$ matrix obtained by deleting the 1st row and the ith column.

Theorem 3. *The definitions 1 and 2 are the same.*

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[1] In this document, elements of the set are integers, though this proof can easily extend to any commutative ring.

Proof. We prove this by induction on n.

Base case: When $n = 1$, S_1 has only 1 element. Hence, by both definitions, $\det(A) = a_{11}$.

Induction step: We assume that the two definitions lead to the same determinant value for square matrices of size $(n-1)$.

In the permutation based definition, consider the term corresponding to the permutation $\mu \in S_n$. It is given by $\operatorname{sgn}(\mu) \cdot a_{1\mu(1)} a_{2\mu(2)} a_{3\mu(3)} \dots a_{n\mu(n)}$. Observe that the minor based definition of determinant will have a term with $a_{1\mu(1)} a_{2\mu(2)} a_{3\mu(3)} \dots a_{n\mu(n)}$. We only have to prove that the sign of that term in the minor based expression is equal to $\operatorname{sgn}(\mu)$. In the minor based definition, we set $i = \mu(1)$. Since A_{1i} is a square matrix of size $(n-1)$, the two definitions lead to the same value of determinant . Let us call the permutation in A_{1i} that leads to the term $a_{2\mu(2)} a_{3\mu(3)} \dots a_{n\mu(n)}$ in A_{1i} as permutation α. We map it to a permutation over the set $\{2, 3, \dots n\}$ by increasing the value of each element by 1 and call it permutation β. We can further map the range to the range $\{1, 2, \dots, n\} \setminus \{i\}$. Let us call this bijection γ. We do the following to get the range:

$$\gamma(j) = \begin{cases} \beta(j) - 1 & \text{if } \beta(j) \le i, \\ \beta(j) & \text{if } \beta(j) > i, \end{cases}$$

We map 1 to i and j to $\gamma(j)(j \in \{2, 3, \dots, n\})$ This is exactly the permutation we get by mapping $1 \longrightarrow i, 2 \longrightarrow \mu(2), \dots, n \longrightarrow \mu(n)$. This is our permutation μ.

Mathematically, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & n-1 & n \\ 1 & i_2 & i_3 & \cdots & i_{i-1} & i_i & i_{i+1} & \cdots & i_{n-1} & i_n \end{pmatrix}$$

where $i_2 = \beta(2), \dots$. Let

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & n-1 & n \\ i & 1 & 2 & \cdots & i_{i-2} & i_{i-1} & i_{i+1} & \cdots & i_{n-1} & i_n \end{pmatrix}$$

Then applying σ and then π gives

$$\pi \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & i-1 & i & i+1 & \cdots & n-1 & n \\ i & i'_2 & i'_3 & \cdots & i'_{i-1} & i'_i & i'_{i+1} & \cdots & i'_{n-1} & i'_n \end{pmatrix} = \mu$$

where

$$i'_j = \begin{cases} i_j - 1 & \text{if } i_j \le i, \\ i_j & \text{if } i_j > i, \end{cases}$$

Hence, $\operatorname{sgn}(\pi) \cdot \operatorname{sgn}(\sigma) = \operatorname{sgn}(\mu)$. Now, $\pi = (i, i-1, \dots, 3, 2, 1)$. Hence, $\operatorname{sgn}(\pi) = (-1)^{i-1}$. Hence, $\operatorname{sgn}(\mu) = (-1)^{i-1} \operatorname{sgn}(\sigma)$. ☐