1 Introduction

There are more than one definitions of the determinant of a matrix. In this document, we will reduce one definition to another. Though I knew both the definitions for a long time, this time I could not recall why the definitions were the same. Hence, after I got the (pretty simple) answer I decided to TeX it.

A determinant for an $n \times n$ matrix $A = [a_{ij}], 1 \leq i, j \leq n$ over some commutative ring\footnote{In this document, elements of the set are integers, though this proof can easily extend to any commutative ring.} is defined as follows:

**Definition 1** (Permutation based definition of determinant).

$$
\det(A) = \sum_{\sigma \in S_n} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},
$$

where $S_n$ is the symmetric group of order $n$.

A recursive definition of the determinant is given by:

**Definition 2** (Minor based definition of determinant).

$$
\det(A) = \begin{cases} 
\sum_{i=1}^{n} (-1)^{i-1} a_{1i} \cdot \det(A_{1i}) & \text{if } n > 1, \\
a_{11} & \text{if } n = 1,
\end{cases}
$$

where $A_{1i}$ is the $(n - 1) \times (n - 1)$ matrix obtained by deleting the 1st row and the $i$th column.

**Theorem 3.** The definitions 1 and 2 are the same.
Proof. We prove this by induction on $n$.

Base case: When $n = 1$, $S_1$ has only 1 element. Hence, by both definitions, $\det(A) = a_{11}$.

Induction step: We assume that the two definitions lead to the same determinant value for square matrices of size $(n - 1)$.

In the permutation based definition, consider the term corresponding to the permutation $\mu \in S_n$. It is given by $\text{sgn}(\mu) \cdot a_{1\mu(1)}a_{2\mu(2)}a_{3\mu(3)}\ldots a_{n\mu(n)}$.

Observe that the minor based definition of determinant will have a term with $a_{1\mu(1)}a_{2\mu(2)}a_{3\mu(3)}\ldots a_{n\mu(n)}$. We only have to prove that the sign of that term in the minor based expression is equal to $\text{sgn}(\mu)$. In the minor based definition, we set $i = \mu(1)$. Since $A_{1i}$ is a square matrix of size $(n - 1)$, the two definitions lead to the same value of determinant. Let us call the permutation in $A_{1i}$ that leads to the term $a_{2\mu(2)}a_{3\mu(3)}\ldots a_{n\mu(n)}$ as permutation $\alpha$. We map it to a permutation over the set $\{2, 3, \ldots, n\}$ by increasing the value of each element by 1 and call it permutation $\beta$. We further map the range to the range $\{1, 2, \ldots, n\} \setminus \{i\}$. Let us call this bijection $\gamma$. We do the following to get the range:

$$
\gamma(j) = \begin{cases} 
\beta(j) - 1 & \text{if } \beta(j) \leq i, \\
\beta(j) & \text{if } \beta(j) > i,
\end{cases}
$$

We map 1 to $i$ and $j$ to $\gamma(j) (j \in \{2, 3, \ldots, n\})$. This is exactly the permutation we get by mapping $1 \rightarrow i, 2 \rightarrow \mu(2), \ldots, n \rightarrow \mu(n)$. This is our permutation $\mu$.

Mathematically, let

$$
\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & i - 1 & i & i + 1 & \cdots & n - 1 & n \\
i_2 & i_3 & \cdots & i_{i-1} & i_i & i_{i+1} & \cdots & i_{n-1} & i_n \end{pmatrix}
$$

where $i_2 = \beta(2), \ldots$. Let

$$
\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & i - 1 & i & i + 1 & \cdots & n - 1 & n \\
i & 1 & 2 & \cdots & i_{i-2} & i_{i-1} & i_{i+1} & \cdots & i_{n-1} & i_n \end{pmatrix}
$$

Then applying $\sigma$ and then $\pi$ gives

$$
\pi \cdot \sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & i - 1 & i & i + 1 & \cdots & n - 1 & n \\
i & i_2 & i_3 & \cdots & i_{i-1} & i_i' & i_{i+1} & \cdots & i_{n-1}' & i_n' \end{pmatrix} = \mu
$$

where

$$
i_j' = \begin{cases} 
i_j - 1 & \text{if } i_j \leq i, \\
i_j & \text{if } i_j > i,
\end{cases}
$$

Hence, $\text{sgn}(\pi) \cdot \text{sgn}(\sigma) = \text{sgn}(\mu)$. Now, $\pi = (i, i - 1, \ldots, 3, 2, 1)$. Hence, $\text{sgn}(\pi) = (-1)^{i-1}$. Hence, $\text{sgn}(\mu) = (-1)^{i-1} \text{sgn}(\sigma)$. \qed