Segment Trees

League of Programmers

ACA, IIT Kanpur
Outline

1. Segment Trees

2. Problems
A Simple Problem

Problem Statement
We have an array $a[0 \ldots n-1]$. 
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We should be able to

1. Find the sum of elements \( l \) to \( r \)
2. Change in the value of a specified element of the array \( a[i] = x \)
A Simple Problem

Possible Solutions

Naive one: Go on from l to r and keep on adding and update the element when you get an update request.
Running time: O(n) to sum and O(1) to update

Store sum from start to i at the i-th index in another array.
Running time: O(1) to return sum, O(n) to update

This works well if the number of query operations are large and very few updates.

What if the number of query and updates are equal?
Can we perform both the operations in O(log n) time once given the array?
Possible Solutions

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A Simple Problem

Possible Solutions

- Naive one: Go on from 1 to r and keep on adding and update the element when you get a update request. Running time: \( O(n) \) to sum and \( O(1) \) to update
- Store sum from start to i at the \( i^{th} \) index in an another array. Running time: \( O(1) \) to return sum, \( O(n) \) to update
A Simple Problem

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Segment Trees

Problems

Representation of the tree

Leaf nodes are the elements in the array. Each internal node represents some merging of the leaf nodes.

Number each node in the tree level by level. Observe that for each node the left child is $2i$ and right child is $2i+1$.

For each node the parent is $i/2$.

Just use an array to represent the tree, operate on indices to access parents and children.

Note: An important feature of the tree segments is that they use linear memory: standard tree segments requires about $4n$ memory elements to work on an array of size $n$. 
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- Note: An important feature of the tree segments is that they use linear memory: standard tree segments requires about $4n$ memory elements to work on an array of size $n$. 
We start with a segment \([0 \ldots n-1]\). and every time we divide the current period of two (if it has not yet become a segment of length), and then calling the same procedure on both halves, and for each such segment we store the sum on it. In other words, we calculate and remember somewhere sum of the elements of the array, i.e., segment \(a[0 \ldots n-1]\). Also calculate the amount of the two halves of the array: \(a[0 \ldots n/2]\) and \(a[n/2+1 \ldots n-1]\). Each of the two halves, in turn, divide in half and count the amount to keep them, then divide in half again, and so on until it reaches the current segment length 1.

The number of vertices in the worst case is estimated at:

\[
\frac{n}{1} + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \cdots + 1 < 2^n
\]

The height of the tree is the value of the segments \(O(\log n)\).
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$$n + n/2 + n/4 + n/8 + \cdots + 1 < 2n$$

- The height of the tree is the value of the segments $O(\log n)$. 
Building the tree

From the bottom up: first write the values of the elements $a[i]$ at the corresponding leaves of the tree, then on the basis of these values to calculate the nodes of the previous level as the sum of the two leaves, then similarly calculate values for one more level, etc. Convenient to describe the operation recursively.

Time to do this? $O(n)$ since each node in the tree is visited once and uses only a max of 2 nodes (already computed) for computation.
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- Time to do this?
  $O(n)$ since each node in the tree is modified once and uses only a max of 2 nodes (already computed) for computation.
The input is two numbers l and r. And we have the time $O(\log n)$. Calculate the sum of the segment $a[l...r]$. Can be done recursively. If your range is within the segment completely, return the value at that node. If it's completely out of range, return 0 or null. If it's in one of the child, query on that child. If it's in both the child, do query on both of them.
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Query Request

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Implementation

Query Request

Pseudocode

```c
query(node, l, r) {
    if range of node is within l and r
        return value in node
    else if range of node is completely outside l and r
        return 0
    else
        return sum(query(left-child, l, r), query(right-child, l, r))
}
```

But this does not look like O(log n)

It is. At any level of the tree, the maximum number of segments that could call our recursive function when processing a request is 4. So, only O(log n) running time.
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- It is.
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- So, only O(log n) running time.
### Implementing a Segment Tree

**Renewal Request**

Given an index `i` and the value of `x`, what to do?

- Update the nodes in the tree so as to confirm to the new value `a[i] = x` in `O(log n)`.

- How many nodes and what nodes will be affected?
  - The nodes from the `i`th leaf node to the way up to the root of the tree.

Then it is clear that the update request can be implemented as a recursive function: it sends the current node of the tree lines, and this function performs a recursive call from one of his two sons (the one that contains the position `i` in its segment), and after that - counts the value of the sum in the current node in the same way as we did in the construction of a tree of segments (i.e., the sum of the values for the two sons of the current node).
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- Given an index \( i \) and the value of \( x \). What to do?
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- How many nodes and what nodes will be affected? The nodes from $i^{th}$ leaf node to the way up to the root of the tree.
- Then it is clear that the update request can be implemented as a recursive function: it sends the current node of the tree, and this function performs a recursive call from one of his two sons (the one that contains the position $i$ in its segment), and after that - counts the value of the sum in the current node in the same way as we did in the construction of a tree of segments (ie, the sum of the values for the two sons of the current node).
You are given an array. You have queries which ask for the maximum element in the range l to r. And you have updates which increase the value of a particular element in the array with some value val.

But seriously, how to code?

Let's look at a code.
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Links: