Zeros of the Riemann Zeta-Function

CS397 Special Topics in Computer Science
Project Report

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Introduction

The Riemann zeta-function $V(s)$ is defined by the (Dirichlet) series

$$V(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for Re(s) > 1,}$$

and by analytic continuation otherwise.

Although zeta-function $V(s)$ was first studied as a function of real s > 1 by Euler who used it to obtain an “analytic proof” of the infinitude of primes and proved, among other things, that $V(2) = \frac{p^2}{6}$ and $V(4) = \frac{p^4}{90}$. But it was the Genius of Riemann who extended the zeta-function to the entire complex plane and found intimate connection between the zeta-function $V(s)$, esp. its zeros, and the distribution of prime numbers. In 1859, Riemann published his epoch-making 8-page paper “Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse” (On the Number of Primes Less Than a Given Magnitude) where he proved a number of theorems about the properties of the zeta-function $V(s)$ and made a number of statements whose proofs had to wait for the developments in theory of functions of a complex variable and lead to proof of the Prime Number Theorem (PNT) by Hadamard and de la Vallée-Poussin in 1896. Although many proofs of the PNT have been obtained but all of them, except for the elementary proofs by Selberg(1949) and Erdös(1949) and Daboussi(1984), use the properties of Riemann’s zeta-function, in particular the non-vanishing of the zeta-function $V(s)$ on the line Re(s) = 1. All the statements made by Riemann about his zeta function were subsequently proved (by Hadamard, von Mangoldt and others) except for the Riemann Hypothesis about the complex zeros of the zeta-function which has defied any attempts of proof for nearly last 150 years and is presently considered the most important unsolved problem in whole of mathematics. It has achieved the kind of stature never possessed by any conjecture in the history of mathematics and its importance cannot be over-emphasized, I just quote Peter Sarnak (of Princeton University) from K. Sabbagh's book “Dr. Riemann’s Zeros”:

"The Riemann Hypothesis is the central problem and it implies many, many things. One thing that makes it rather unusual in mathematics today is that there must be over five hundred papers - somebody should go and count - which start "Assume the Riemann Hypothesis", and the conclusion is fantastic. And those [conclusions] would then become theorems...With this one solution you would have proven five hundred theorems or more at once."

In this special topics course, I've studied some of the literature on the theory of the Riemann zeta-function $V(s)$ and tried to get an idea about the results known presently. As we shall see, the most important thing to know is the location of the complex zeros of the zeta-function $V(s)$. It has been known since long that the zeta-function $V(s)$ has zeros on the negative real-axis at s = -2, -4, -6,……… (the negative even integers), called trivial zeros, and all other zeros of $V(s)$ are complex, lie in the region 0 < Re(s) <1 ("the critical strip"), and are situated symmetrically about the real axis and also the line Re(s) = ½ ("the critical line"). The Riemann Hypothesis is the statement that all the complex zeros (also called the non-trivial zeros) of the zeta-function $V(s)$ actually lie on the critical line. Since Riemann Hypothesis is much too important problem and has been defying all the attempts to prove it, many individuals and groups have been collecting as much data as possible to get evidence in its favor (or, possibly find a counter-example to the Riemann Hypothesis, though it will be a Nightmare come true…). There have been many efficient numerical methods and algorithms developed to calculate the complex zeros of the zeta-function, some of them specifically suited for the calculation of the zeros of the zeta function $V(s)$ that lie on the critical line. At the same time theory and methods have been developed to ensure that there are no complex zeros of the zeta function $V(s)$ off the critical line in a rectangular region of the form | Im s | ≤ T where the zeros on the critical line have been calculated, and thus they verify the validity of the Riemann Hypothesis in a rectangular region of the same form. As of now, an organization called ZetaGrid founded by Sebastian Wedeniwski of IBM has calculated first 385 billion complex zeros of the zeta-function $V(s)$ and has found that not only all these zeros lie on the critical line but also they are all simple zeros. The main theoretical ingredient in all these computations is the so-called Riemann-Siegel formula which was essentially devised by Riemann and found by Carl Siegel in 1932 through his study of an unpublished manuscript of Riemann. We discuss, but very briefly, some aspects of these computations also.

In this report, I've presented the part of the theory of the Riemann Zeta-Function that I've studied in this course, mostly without proofs (although appropriate references have been provided) so that it can give an overview of the results known without much effort on the part of the reader if the reader is ready to take some things on faith. I should also mention that I've been able to study only the most elementary part of the vast theory developed on and for the subject of the zeta-function. Needless to mention, some new (and most of them are extremely difficult) branches of mathematics are being developed in hope to solve Riemann Hypothesis, some of the most notable of these efforts being 1982 Fields Medalist Alain Connes' work on Non-Commutative Geometry and Michel L. Lapidus and Machiel van Frankenhuysen's work on Fractal Geometry.

Chapter 1
The Riemann Zeta-Function

1.1 Analytic Continuation of the Zeta-Function

Definition The Riemann zeta-function $V(s)$ is defined by the series

$$V(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for Re(s) > 1} \qquad (1).$$

Following the tradition of Riemann, we write s = σ+it with σ and t real, i^2= -1. The series in (1) converges absolutely for Re(s) = σ > 1 and uniformly in any region of the form Re(s) = σ ≥ s_0 > 1, and hence, by a theorem of Weierstrass, represents a holomorphic (analytic) function of s in the region Re(s) = σ > 1.

Theorem 1.1.1 (Euler Product for the Riemann Zeta-Function) For Re(s) > 1

$$V(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathrm{P}} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (2).$$

where the product is taken over all the primes.

The identity follows from the absolute convergence of the series and the product and the Fundamental Theorem of Arithmetic. We could well have taken (2) as the definition of the Riemann zeta-function $V(s)$ and obtained (1) as a theorem.

As it stands the zeta-function $V(s)$ is defined only in the left half plane Re(s) > 1 of the complex s-plane. Riemann extended the zeta-function $V(s)$ to the entire complex s-plane, i.e., he obtained an analytic (meromorphic to be precise) function defined on the whole complex plane and agrees with the function above in the region where it is already defined (i.e. the region Re(s) > 1).The theta-series plays an important role in the analytic continuation of the zeta-function.

Definition For an arbitrary complex number α and a complex number τ with Re(τ) > 0 we define the theta-series θ(τ,α) by

$$\theta(\tau,\alpha) = \sum_{n=-\infty}^{\infty} \exp(-pt\,(n+a)^2 .$$

Theorem 1.1.2 (Functional Equation of the Theta-Series) We have,

$$\theta(1/\tau,\alpha) = \sqrt{t}\ \sum_{n=-\infty}^{\infty} \exp(-ptn^2 + 2pina) + \sqrt{t}\ \exp(-pa^2/t)\,\theta(\tau,-ia/t) .$$

In particular, for real x > 0 and θ(x) := θ(x,0) = $\sum_{n=-\infty}^{\infty}\exp(-ptn^2)$, we have,

$$\theta(1/x) = \sqrt{x}\,\theta(x) \qquad (3).$$

For a proof see [KaVo], or [TiHB] where many different proofs are given.

Theorem 1.1.3 (Continuation of the Riemann Zeta-Function) For Re(s) > 1,

$$p^{-s/2}\Gamma(s/2)V(s) = \frac{1}{s(s-1)} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})w(x)dx \qquad (4).$$

where ω(x) := $\sum_{n=1}^{\infty}\exp(-pxn^2)$ = (θ(x) – 1)/2.

Proof: We will present only the outline of the proof as details can be easily filled in or looked into any text on the Riemann zeta-function. For Re(s) > 0, we have the integral representation for the gamma function:

$$\Gamma(s) = \int_0^{\infty} u^{s-1}e^{-s}du\,.$$

Replace s by s/2 and then apply a change of variable u = pn^2x to get

$$\Gamma(s/2) = n^s p^{s/2}\int_0^{\infty} x^{s/2-1}e^{-pn^2x}dx\,.$$

Transpose $n^s p^{s/2}$ on the left and then sum over all natural numbers n, and note that we can change the order of summation and integration on the right because of absolute convergence, to get

$$p^{-s/2}\Gamma(s/2)V(s) = \int_0^{\infty} x^{\frac{s}{2}-1}w(x)dx\,.$$

Divide the integral on the right into one from 0 to 1 and other from 1 to ∞ and apply change of variable x→1/x in the first. Note that the functional equation for θ(x) transforms to that for ω(x) as 2ω(x)+1 = $\frac{1}{\sqrt{x}}$(2ω(1/x)+1) to get (4).

Let us note that ω(x) = O(exp(-πx)) as x → +∞, so that the integral on the right converges absolutely and uniformly in a region of the form Re(s) > K for any real number K (K maybe negative) and hence represents a holomorphic function of s defined over the entire complex plane. Also, the gamma function Γ(s) has simple poles at s = 0,-1, -2, -3,… (so that Γ(s/2) has simple poles at s = 0, -2, -4, -6,…) with its residue at s = 0 being $\operatorname{Res}_{s=0}\Gamma(s) = \lim_{s\to 0} s\Gamma(s) = \lim_{s\to 0}\Gamma(s+1) = \Gamma(1) = 1$.

Note also the particular value Γ(1/2) = $\sqrt{p}$.

Hence, 1/ Γ(s/2) is an entire function with simple zeros at s = 0, -2, -4, -6,………

Therefore, the expression on the right in the equation

$$V(s) = \frac{p^{s/2}}{\Gamma(s/2)}\left(\frac{1}{s(s-1)} + \int_1^{\infty}(x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})w(x)dx\right) \qquad (5).$$

makes sense for all values of s ≠ 1 (at s =1 it has a simple pole with residue 1) and agrees with the zeta-function ζ(s) as defined by (1) in the region Re(s) > 1. Hence, it gives analytic continuation of the zeta-function $V(s)$ to the entire complex plane. The zero of 1/Γ(s/2) at s =0 cancels the pole of the term 1/s(s-1), while the zeros of 1/Γ(s/2) at s = -2, -4, -6,……… give rise to the trivial zeros of ζ(s), because of the pole of the term 1/s(s-1) at s = 1 the zeta-function ζ(s) has a simple pole at s = 1 with its residue at s =1 being

$$\lim_{s\to 1}(s-1)V(s) = \lim_{s\to 1}\frac{p^{s/2}}{\Gamma(s/2)}\left(\frac{1}{s(s-1)} + \int_1^\infty (x^{\frac{s}{2}-1} + x^{-\frac{s}{2}-\frac{1}{2}})w(x)dx\right) = \frac{p^{1/2}}{\Gamma(1/2)} = 1 .$$

Note that the right side in (4) does not change if we replace s by 1-s. If we define

$$x(s) := \frac{1}{2}s(s-1)p^{-s/2}\Gamma(s/2)V(s) \qquad (6).$$

then ξ(s) is an entire function and satisfies the functional equation

$$\xi(s) = \xi(1\text{-}s). \qquad (7).$$

The functional equation for the Riemann zeta-function $V(s)$ is

$$p^{-s/2}\Gamma(s/2)V(s) \;=\; p^{-(1-s)/2}\Gamma((1-s)/2)V(1-s) \qquad (8).$$

From the convergent Euler product expression for $V(s)$ given by (2), we see that the zeta-function ζ(s) does not vanish in the region Re(s) > 1. Hence by (6) the function ξ(s) does not vanish for Re(s) > 1 and then by (7) ξ(s) does not vanish in the region Re(s) < 0 either. Hence all its zeros lie in the strip 0 ≤ Re(s) ≤ 1 and they are the zeros of the $V(s)$ in that strip. By (8) then the only zeros of the $V(s)$ in the region Re(s) < 0 are those arising due to poles of Γ(s/2), i.e., the “trivial” zeros at the negative even integers; and all the non-trivial zeros of $V(s)$ are, in fact, the zeros of the function ξ(s). Hadamard and de la Vallée Poussin proved in 1896 that the zeta-function $V(s)$ does not vanish on the line Re(s) = 1 (and hence by (7) on the line Re(s) = 0 as well) thereby proving the Prime Number Theorem.

1.2 Entire Functions

Since ξ(s) is an entire function, we can apply all the results from the theory of entire functions to obtain information about ξ(s) and hence about $V(s)$. We collect here some of the relevant facts about entire functions and the gamma function that we will be useful to us later in proving some results about the zeros of the zeta-function $V(s)$.

Theorem 1.2.1 (Weierstrass) There exists an entire function f(z) with arbitrarily prescribed zeros $\{r_n\}$ provided that in case of infinitely many zeros $|r_n| \to \infty$.
(if some number in the sequence of zeros is repeated then the corresponding number will be a zero of f(z) with appropriate multiplicity.) Every entire function with precisely these zeros can be expressed in the form

$$f(z) = z^m e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{r_n}\right) e^{\left(\frac{z}{r_n}\right) + \frac{1}{2}\left(\frac{z}{r_n}\right)^2 + \frac{1}{3}\left(\frac{z}{r_n}\right)^3 + \ldots + \frac{1}{m_n}\left(\frac{z}{r_n}\right)^{m_n}} \quad (9).$$

where m is the multiplicity of the zero at z =0 (m ≤ 0), g(z) is an entire function and m_n are integers and the product is taken over non-zero zeros of f(z). (For example we could take m_n = n-1 to get one such function.)

Definition An entire function f(z) is said to have finite order (of growth) if there exists a > 0 and r_a > 0 such that $\underbrace{\max}_{|z|=r} |f(z)| < e^{r^a}$ for r > r_a. In this case, the order of f(z) is defined to be α := inf { a **|** ∃ r_a > 0 such that $\underbrace{\max}_{|z|=r} |f(z)| < e^{r^a}$ for r > r_a }.

Definition A sequence $\{r_n\}$ is said to have finite exponent of convergence if there exists b > 0 such that $\sum |r_n|^{-b}$ converges. In this case, the exponent of convergence of $\{r_n\}$ is defined to be β : = inf { b **|** $\sum |r_n|^{-b}$ converges}.

Theorem 1.2.2 (Hadamard) Let f(z) be an entire function not vanishing at zero with zeros $\{r_n\}$. If there exists an integer p ≥ 0 such that $\sum |r_n|^{-(p+1)}$ converges then (m =0, of course, and) we can take m_n = p for all n in (9).

Theorem 1.2.3 Let f(z) be an entire function of finite order α not vanishing at zero $\{r_n\}$ being its sequence of zeros arranged in non-decreasing order of modulus. Then the sequence $\{r_n\}$ has a finite exponent of convergence β ≤ α and f(z) can be expressed in the form

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} \left(1 - \frac{z}{r_n}\right) e^{\left(\frac{z}{r_n}\right) + \frac{1}{2}\left(\frac{z}{r_n}\right)^2 + \frac{1}{3}\left(\frac{z}{r_n}\right)^3 + \ldots + \frac{1}{p}\left(\frac{z}{r_n}\right)^{p}} \quad (10).$$

where p ≥ 0 is the smallest integer such that $\sum |r_n|^{-(p+1)}$ converges, and g(z) is a polynomial with deg g ≤ α. In fact, α = max {deg g, β}. Also, if the series $\sum |r_n|^{-1}$ converges then there exists a constant c > 0 such that $\underbrace{\max}_{|z|=r} |f(z)| < e^{c(r+1)}$ for all r.

1.3 The Gamma Function

Definition The Euler gamma-function Γ(s) is defined through the equality

$$\frac{1}{\Gamma(s)} = s e^{g} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \quad (11).$$

where g is Euler's constant, i.e., $g := \lim_{N \to \infty} \left(\sum_{n=1}^{N} n^{-1} - \log N\right)$ = 0.57721566490…

Equivalently, we can write

$$\Gamma(s) = \frac{1}{se^{\gamma}} \prod_{n=1}^{\infty} e^{s/n}\left(1+\frac{s}{n}\right)^{-1} = \frac{1}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1} \qquad (12).$$

$$\Gamma(s) = \lim_{n\to\infty} \frac{n!\,n^{s}}{s(s+1)(s+2)...(s+n)} \qquad (13).$$

We can start with definition (11) or (12) which clearly show that 1/Γ(s) has simple zeros at s = 0, -1, -2, -3,…(and hence Γ(s) has simple poles at s = 0, -1, -2, -3,…) and derive the integral formula for Γ(s) in the region Re(s) > 0; or , alternatively, we could start with the usual definition of Γ(s) through the integral formula and obtain analytic continuation of the function 1/Γ(s) to get an entire function of order one which will have the product development (11) by theorem 1.2.3. We now list some properties of the gamma function.

Theorem 1.3.1 (Functional Equation of the Gamma Function) For all s,

$$\Gamma(s+1) = s\,\Gamma(s) \qquad (14).$$

This with Γ(1) = 1 gives us for any natural number n, Γ(n) = (n-1)!

Theorem 1.3.2 (Addition Formula) For s not an integer,

$$\Gamma(s)\,\Gamma(1-s) = \pi / \sin \pi s \qquad (15).$$

Theorem 1.3.3 (Multiplication Formula) For alls and any natural number m,

$$\Gamma(s)\Gamma\left(s+\frac{1}{m}\right)\Gamma\left(s+\frac{2}{m}\right)..\Gamma\left(s+\frac{m-1}{m}\right) = (2p)^{(m-1)/2} m^{1/2-ms}\Gamma(ms) \qquad (16).$$

Theorem 1.3.4 (Integral Formula) For s with Re(s) > 0,

$$\Gamma(s) = \int_0^{\infty} e^{-u} u^{s-1} du \qquad (17).$$

Theorem 1.3.5 (Stirling's Formula) For δ > 0 and -π+δ ≤ arg s ≤ π-δ,

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \log\sqrt{2p} + \int_0^{\infty}\frac{(1/2)-\{u\}}{u+s}du \qquad (18).$$

$$\log\Gamma(s) = \left(s-\frac{1}{2}\right)\log s - s + \log\sqrt{2p} + O\left(\frac{1}{|s|}\right) \qquad (19).$$

Differentiating (17) we get the following relations for logarithmic derivative of Γ(s).

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \int_0^{\infty}\frac{(1/2)-\{u\}}{(u+s)^2}du \qquad (20).$$

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log s - \frac{1}{2s} + O\left(\frac{1}{|s|^2}\right) = \log s + O\left(\frac{1}{|s|}\right) \qquad (21).$$

where the constants in the O-terms depend only on δ.

1.4 Continuation of ζ (s) to the Region Re(s) > 0

Theorem 1.4.1 (Abel Summation Formula) Let f(x) be a complex-valued function of a real variable continuously differentiable on [a,b], c_n a sequence of complex numbers and define C(x) := $\sum_{n\le x} c_n$. Then

$$\sum_{a<n\le b} c_n f(n) = C(b)f(b) - C(a)f(a) - \int_a^b C(x)f'(x)dx \qquad (22).$$

Proof: Using Stieltjes integral and applying integration by parts we get LHS = $\int_{a+0}^{b+0} f(x)dC(x)$ =RHS.

Theorem 1.4.2 (Euler Summation Formula) Let f(x) be a complex-valued function of a real variable continuously differentiable on [a,b]. Then

$$\sum_{a<n\le b} f(n) = \int_a^b f(x)dx + \int_a^b \{x\} f'(x)dx + \{a\}f(a) - \{b\}f(b) \qquad (23).$$

$$= \int_a^b f(x)dx + \int_a^b r_1(x) f'(x)dx + r_1(a)f(a) - r_1(b)f(b) \qquad (24).$$

where {x} := x – [x] and ρ_1(x) := x – [x] – ½ = {x} – ½.

For a proof see [Ap] or [KaVo].

Theorem 1.4.3 (Continuation of ζ (s) to the Region Re(s) > 0) For Re(s) > 0 and a natural number N we have,

$$V(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2}N^{-s} + s\int_N^\infty \frac{r_1(u)du}{u^{s+1}} \qquad (25).$$

In particular, taking N=1 we get, for Re(s) > 0,

$$V(s) = \frac{s}{s-1} - \frac{1}{2} + s\int_N^\infty \frac{r_1(u)du}{u^{s+1}} \qquad (26).$$

Proof: For Re(s) > 1, divide the sum in (1) into two parts one from 1 to N another from N to ∞. Choosing M > N, applying Euler summation formula with f(x) = x^{-s} to the sum $\sum_{N+\frac{1}{2}<n\le M+\frac{1}{2}}^{M} \frac{1}{n^s}$ and letting M → ∞ we get (25). Since the integral in the right side of (25) converges absolutely for Re(s) > 0 and uniformly in any region of the form Re(s) = σ ≥ σ_0 > 0 and, therefore, represents a holomorphic function in that region and must equal $V(s)$ there by uniqueness of analytic continuation.

1.5 x(s) as an Entire Function

Theorem 1.5.1 ξ(s) is an entire function of order 1.
Proof: Since ξ(s) = ξ(1-s), we need to bound ξ(s) only for Re(s) ≥ ½.
For Re(s) > 0, from (26) we have $|s(s\text{-}1)\, V(s)| = O(|s|^3)$ and by (19) we have $|\Gamma(s)| \le (c_1 \exp(|s| \log|s|))$. Since $|\pi^{-s/2}| \le \exp(c_2|s|)$, it follows that
$x(s) = \frac{1}{2} s(s-1) p^{-s/2} \Gamma(s/2) V(s)$ is a function of order at most 1. But when s→ +∞ (i.e. approaches ∞ increases through positive real values) then log Γ(s) ~ s log s and hence log ξ(s) ~ s log s as well and hence the order of ξ(s) is equal to 1.

Since $V(s)$ has a simple pole with residue 1 at s =1, we get from (6) and (7)

ξ(0) = ξ(1) = $\lim_{s \to 1} \frac{1}{2} s(s-1) p^{-s/2} \Gamma(s/2) V(s) = \lim_{s \to 1} (s-1) V(s) \frac{1}{2} s p^{-s/2} \Gamma(s/2)$

$= 1 \times \frac{1}{2} \times 1 \times p^{-1/2} \Gamma(1/2) =$ ½.

By theorem 1.2.3, then, ξ(s) can be represented in the form

$$x(s) = \frac{1}{2} s(s-1) p^{-s/2} \Gamma(s/2) V(s) = e^{A+Bs} \prod_{n=1}^{\infty} \left(1 - \frac{s}{r_n}\right) e^{s/r_n} \qquad (27).$$

where $r_n = b_n + i\, g_n$ are zeros of ξ(s), i.e., non trivial zeros of $V(s)$, arranged in the order of increasing modulus of the imaginary part (and when two or more zeros have the imaginary parts with the same modulus then they are arranged in an arbitrary manner) and the series $\sum_{n=1}^{\infty} \frac{1}{|r_n|^{1+e}}$ converges for any $e > 0$.And since log Γ(s) ~ s log s when s→ +∞ there does not exists a constant c > 0 such that $\underbrace{\max}_{|s|=r} |x(s)| < e^{c(r+1)}$ for all r and hence the series $\sum_{n=1}^{\infty} \frac{1}{|r_n|}$ diverges which implies that there is an infinity of zeros of $x(s)$ and hence the non-trivial zeros of ζ(s). Also, from (5) and (6), it follows that

$$x(\bar{s}) = \overline{x(s)} \qquad (28).$$

From (7) and (28), we see that if $r_n = b_n + i\, g_n$ is a zero of ξ(s), or, equivalently, is a non-trivial zero of $V(s)$, then so are $\overline{r_n}$, $1 - r_n$, and $1 - \overline{r_n}$. Next, for Re(s) > 1,

$$\left(1 - 2^{1-s}\right) V(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} = 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \ldots$$

and the last series converges uniformly, albeit not absolutely, in compact subsets of the region Re(s) > 0 and thus defines an analytic function in that region which must equal $\left(1 - 2^{1-s}\right) V(s)$ in that region by virtue the of uniqueness of the analytic continuation. Since for (real) s >0,

$$1-\frac{1}{2^s}+\frac{1}{3^s}-\frac{1}{4^s}+\ldots = \left(1-\frac{1}{2^s}\right)+\left(\frac{1}{3^s}-\frac{1}{4^s}\right)+\ldots>0\,,$$

we see that $V(s)\neq 0$ for (real) s >0, i.e., $V(s)$ has no real zeros in the critical strip so that the zeros $r_n = b_n + i\, g_n$ in (27) are all complex.

If we take the logarithmic derivative on both sides of (27), and substitute that of (12) (with s replaced by s/2), we get the following result.

Theorem 1.5.2 The zeta function satisfies the relation

$$\frac{V'(s)}{V(s)} = -\frac{1}{s-1}+\sum_{n=1}^{\infty}\left(\frac{1}{s-r_n}+\frac{1}{r_n}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{s+2n}-\frac{1}{2n}\right)+c \qquad (29).$$

where r_n are complex zeros of $V(s)$ and c is an absolute constant.

Chapter 2
The Simplest Theorems on the Zeros of The Riemann Zeta-Function z (s)

2.1 Sums over Zeros

We start by fixing our notation: $r = b + i\,g$ and $r_n = b_n + i\,g_n$ will denote a generic zero of $x(s)$, i.e., a complex or non-trivial zero of ζ(s), and nth zero of ξ(s) when zeros are arranged in the order of increasing modulus of their imaginary parts, respectively. We write $s = s + it$, if s has some subscript then s and t will have the same subscripts. c, c_1, c_2,… will denote positive constants; ε, ε_1,…will denote arbitrarily small positive constants. If f and g are functions, f = O(g) or f << g means that there exists a constant c>0 such that |f| ≤ cg. Notations f = $O_{a,b,\ldots}$(g) or f $<<_{a,b,\ldots}$ g mean that f = O(g) or f << g holds for fixed a,b,…
We now list some theorems concerning zeros of the zeta-function ζ(s) without proofs which can be found in [KaVo] or [TiHB]. First we collect all the facts about the zeros of the zeta-function found in the last chapter in a theorem.

Theorem 2.1.1 The negative even integers -2, -4, -6,… are the trivial zeros of the Riemann zeta-function ζ(s). The non-trivial zeros of ζ(s) are all complex, lie in the critical strip 0 < Re(s) < 1, are situated symmetrically about the real axis Im(s) = 0 and the critical line Re(s) = ½, and are the zeros of the entire function ξ(s).

Theorem 2.1.2 For T ≥ 2,

$$\sum_{n=1}^{\infty} \frac{1}{1+(T-g_n)^2} = O(\log T) \qquad (1).$$

Corollary 1 The number of zeros ρ_n of ζ(s) for which T ≤ |Im ρ| ≤ T+1 is O(log T). In particular, the multiplicity of a complex zero ρ_n is bounded by O(log T).

Corollary 2 For T ≥ 2,

$$\sum_{|T-g_n|\ge 1} \frac{1}{(T-g_n)^2} = O(\log T) \qquad (2).$$

Corollary 3 For -1 ≤ σ ≤ 2,

$$\frac{V'(s)}{V(s)} = -\frac{1}{s-1} + \sum_{|T-g_n|\le 1} \frac{1}{s-r_n} + O(\log(|t+2|)) \qquad (3).$$

2.2 Theorems on Zero-free Regions for z (s)

As we shall see, the error term in the Prime Number Theorem depends crucially on the zero-free region for the zeta-function ζ(s) in the critical strip; the bigger the zero-free region, the smaller the error term. Incrementally better zero-free regions have been found over the period of time but the results are still far from satisfying. In this direction, the ultimate dream of mathematicians is to prove the Riemann Hypothesis which states that all the complex zeros of the zeta-function are on the critical line Re(s) = ½ (i.e., ζ(s) ≠ 0 off the critical line).

Theorem 2.2.1 (de la Vallée-Poussin) There exists an absolute constant c > 0 such that ζ(s) ≠ 0 in the following region of the s-plane:

$$\mathrm{Re}(s) = \sigma \geq 1 - \frac{c}{\log(|t|+2)} \qquad (4).$$

Theorem 2.2.2 (Littlewood) There exists an absolute constant c > 0 such that ζ(s) ≠ 0 in the following region of the s-plane:

$$\mathrm{Re}(s) = \sigma \geq 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}} \qquad (5).$$

Theorem 2.2.3 (Vinogradov) There exists an absolute constant c > 0 such that ζ(s) ≠ 0 in the following region of the s-plane:

$$\mathrm{Re}(s) = \sigma \geq 1 - \frac{c}{(\log t)^{2/3}(\log\log t)^{1/3}} \qquad (6).$$

These theorems give error-bounds in the PNT (π(x) ≅ Li(x)) as O(x exp(-c $\sqrt{\log x}$)) O(x exp (-c $\sqrt{\log x \log\log x}$)) and O(x exp (-c $(\log x)^{3/5}(\log\log x)^{-1/5}$)) respectively while the Riemann Hypothesis implies gives an error term of O($\sqrt{x}\log x$).

2.3 Asymptotic Density of the Zeros of z (s)

We denote by N(T) the number of zeros of ζ(s) in the region 0 ≤ Re(s) ≤ 1, 0 ≤ Im(s) ≤ T. There is a very precise formula for N(T) as T → ∞.

Theorem (Riemann-von Mangoldt Formula for N(T)) For T ≥ 2,

$$\mathrm{N(T)} = \frac{T}{2p}\log\frac{T}{2p} - \frac{T}{2p} + O(\log T) \qquad (7).$$

This gives us a feel that as we go up in the critical strip the zeros tend to get denser (like log t “on average”) and in fact it has been proved that for T ≥ T_A≥0, ζ(s) has a root $r = b + i\,g$ with $|g - T|$ < A log log T.

Chapter 3
The Distribution of Prime Numbers

3.1 Preliminaries

Definition. The von Mangoldt function Λ(n) is defined by

$$\Lambda(n) = \begin{cases} \log p ...if ...n = p^k ... for.a.prime.p.and.a.natural.number.k \\ 0,...otherwise. \end{cases} \quad (1).$$

Definition. The prime counting function id denoted by π(x):

$$\pi(x) := \sum_{p \le x} 1 \quad (2).$$

Chebyshev's θ and ψ functions are defined by

$$q(x) := \sum_{p \le x} \log p \quad (3).$$

$$y\ (x) := \sum_{n \le x} \Lambda(n) = \sum_{p,k: p^k \le x} \log p \quad (4).$$

Chebyshev was the first to obtain non-trivial bounds for the prime counting function π(x). In 1852, using combinatorial arguments he proved that

$$0.92 \frac{x}{\log x} \le \pi(x) \le 1.11 \frac{x}{\log x} \quad (5).$$

and that if the limit $\lim_{x \to \infty} \frac{p(x)}{x / \log x}$ exists, it must be equal to 1. The fact that this limit does exist and is equal to 1 is one of the forms of the Prime Number Theorem (abbreviated PNT), and its proof had to wait for the developments in the theory of function of a complex variable for long and was finally given simultaneously by J. Hadamard and C. J. de la Vallée-Poussin in 1896. Hence x / log x is a good approximation to π(x). A better approximation is given by the logarithmic integral:

$$\pi(x) \approx Li(x) := \int_0^x \frac{dt}{\log t} := \lim_{e \to 0} \left(\int_0^{1-e} \frac{dt}{\log t} + \int_{1+e}^x \frac{dt}{\log t} \right) = \int_2^x \frac{dt}{\log t} - 0.4... \quad (6).$$

By repeated application of integration by parts we find that, for any n ≥ 1,

$$Li(x) = \frac{x}{\log x} + \frac{1!x}{(\log x)^2} + \frac{2!x}{(\log x)^3} + ... \frac{(n-1)!x}{(\log x)^n} + O_n\left(\frac{x}{(\log x)^{n+1}} \right) \quad (7).$$

In fact, Vallée-Poussin had proved the Prime Number Theorem in the form

$$\pi(x) = Li(x) + O(x \exp(-c\sqrt{\log x})) \quad (8).$$

for some absolute constant c > 0, which implies that the error in approximating π(x) by Li(x) is $o(x/(\log x)^k)$ for any integer k while that in approximating π(x) by x / log x is $\Omega(x/(\log x)^2)$.

In his 1859 paper, Riemann found a deep connection between the zeta-function ζ(s) and the distribution of the prime numbers. Since π(x) is extremely irregular function and it is comparatively easier to deal with Chebyshev functions, we first transform the statements about π(x) into those about θ(x) and ψ(x).

Theorem 3.1.1 Let E(x) be a function such that $\sqrt{x}$ = O(E(x)) and E(x) / (log x)^2 is an increasing function of x. Then the following are equivalent statements:

(i) ψ(x) = x + O(E(x)).

(ii) θ(x) = x + O(E(x)).

(iii) π(x) = Li(x) + O(E(x) / log x).

Remark. Littlewood proved in 1914 that ψ(x) – x = $\Omega_{\pm}(\sqrt{x}$ log log log x), i.e., there exists a positive constant c > 0 and numbers $x_k \to\infty$, $y_k \to\infty$ such that ψ(x_k) – $x_k \geq$ c $\sqrt{x_k}$ log log log x_k, and ψ(y_k) – $y_k \leq$ c $\sqrt{y_k}$ log log log y_k. Hence, the conditions on E(x) are in fact always valid.

Proof: It is clear that

$$y\ (x) = \sum_{p,k:p^k\leq x}\log p = \sum_{p\leq x}\log p + \sum_{p^2\leq x}\log p + \sum_{p^3\leq x}\log p + ... = q(x) + q(x^{1/2}) + q(x^{1/3}) + ...$$

Since $q(x) \leq y\ (x)$ = O(x), $q(x^{1/2})$ = O(x$^{1/2}$), and right side has less than lox /log 2 terms after the second each of which is O(x$^{1/3}$), we get $y\ (x) = q(x)$ + O($\sqrt{x}$) =$q(x)$ + O(E(x)) i.e. (i) ⇔ (ii). Applying Stieltjes integral we get,

$$p(x) = \int_{2-e}^{x+e}\frac{dq(t)}{\log t} = \frac{q(x)}{\log x} + \int_{2-e}^{x+e}\frac{q(t)dt}{t(\log t)^2} = \frac{x}{\log x} + O\left(\frac{E(x)}{\log x}\right) + \int_{2-e}^{x+e}\frac{t+O(E(t))}{t(\log t)^2}dt$$

$$= \frac{x}{\log x} + \int_{2-e}^{x+e}\frac{dt}{(\log t)^2} + O\left(\frac{E(x)}{\log x}\right) + O\left(E(x)\int_{2-e}^{x+e}\frac{dt}{t(\log t)^2}\right) = Li(x) + O\left(\frac{E(x)}{\log x}\right)$$

since $Li(x) = \frac{x}{\log x} + \int_{2-e}^{x+e}\frac{dt}{(\log t)^2}$ by integration by parts and $\frac{E(x)}{(\log x)^2}$ is increasing.

The converse (iii) ⇒ (i) is proved similarly starting with $q(x) = \int_{2-e}^{x+e}\log t\, dp(t)$.

Theorem 3.1.2 For Re(s) > 1,

$$-\frac{V'(s)}{V(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} \qquad (9).$$

Proof: The identity follows by taking logarithmic derivative of both sides in Euler product for ζ(s), expanding log(1-p^{-s}) into power series and noting that

$$\frac{\Lambda(n)}{n^s} = \begin{cases}\frac{\log p}{kp^{ks}}...if...n = p^k...for.a.prime.p.and.a.natural.number.k\\ 0,...otherwise.\end{cases}$$

3.2 The Explicit Formula

Theorem 3.2.1 (Inversion Formula) Suppose that the series $f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges absolutely for $\sigma = \mathrm{Re}(s) > 1$, $|a_n| \le A(n)$ where $A(x)$ is a positive monotonically increasing function on the positive real line, and also that $\sum_{n=1}^{\infty} \frac{|a_n|}{n^s} = O((s-1)^{-\alpha})$ for some $\alpha > 0$ as $\sigma \to 1+0$. Then the formula

$$\Phi(x) := \sum_{n \le x} a_n = \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s) \frac{x^s}{s} ds + O\left(\frac{x^b}{T(b-1)^{\alpha}} \right) + O\left(\frac{xA(2x)\log x}{T} \right) \qquad (10).$$

holds for any $b_0 \ge b > 1$, $T \ge 1$, $x = N + \frac{1}{2}$ with the constant in O-term depending only on b_0.

The conditions of the above theorem are satisfied with $a_n = \Lambda(n)$, $\Phi(x) = \psi(x)$, $f(s) = \frac{-\zeta'(s)}{\zeta(s)}$, $\alpha = 1$. If we choose $b = 1 + \frac{1}{\log x}$ and apply the bounds obtained in Sec. 2.1 we get the following form of the explicit formula.

Theorem 3.2.2 (The Explicit Formula) For $2 \le T \le x$, we have,

$$\psi(x) := \sum_{n \le x} \Lambda(n) = x - \sum_{|\mathrm{Im}\,\rho| \le T} \frac{x^{\rho}}{\rho} + O\left(\frac{x \log^2 x}{T} \right) \qquad (11).$$

where the sum is taken over the non-trivial zeros ρ of $\zeta(s)$ with $|\mathrm{Im}\,\rho| \le T$.

If we choose appropriate values of T in the interval [2,x] and apply the non-vanishing results of Sec. 2.2, we get PNT with the corresponding error terms. Here, if we try to increase T thereby reducing the O-term, we get bigger error-term in approximating the sum over zeros. It is clear that the tighter the bound on the real part of the non-trivial zeros ρ, the better will be the error bound. This is the place from where the Riemann Hypothesis (RH) derives its major importance.

Theorem 3.2.3 Riemann Hypothesis $\Leftrightarrow$ $\psi(x) = x + O(\sqrt{x}\log^2 x)$. (12).

Proof: Assume RH. Take $T = \sqrt{x}$ in (11), and note that $|x^{\rho}| = \sqrt{x}$ since $\mathrm{Re}\,\rho = ½$.
Then $\left| \sum_{|\mathrm{Im}\,\rho| \le T} \frac{x^{\rho}}{\rho} \right| \le \sum_{|\mathrm{Im}\,\rho| \le T} \frac{|x^{\rho}|}{|\rho|} \le \sqrt{x} \sum_{|\mathrm{Im}\,\rho| \le T} \frac{1}{|\rho|}$ and $\frac{x \log^2 x}{T} = \sqrt{x}\log^2 x$.
Also, since the number of zeros ρ of $\zeta(s)$ with $t \le |\mathrm{Im}\rho| \le t+1$ is $O(\log t)$ for $t \ge 2$,

$$\sum_{|\mathrm{Im}\,\rho| \le T} \frac{1}{|\rho|} = O\left(\sum_{n=2}^{T} \frac{\log n}{n} \right) = O\left(\int_1^T \frac{\log t dt}{t} \right) \text{ and } \int_1^T \frac{\log t dt}{t} = \log^2 T \qquad (13).$$

Conversely, we will assume a weaker statement, viz., ψ(x) = x + O($x^{1/2+\varepsilon}$) for any ε > 0, and prove the RH. For Re(s) > 1, we have,

$$-\frac{V'(s)}{V(s)} = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s} = \int_1^{\infty}\frac{dy\ (x)}{x^s} = \frac{y\ (x)}{x^s}\Bigg|_1^{\infty} + s\int_1^{\infty}\frac{y\ (x)}{x^{s+1}}$$

$$= 0 + s\int_1^{\infty}\frac{x}{x^{s+1}} + s\int_1^{\infty}\frac{y\ (x) - x}{x^{s+1}} = \frac{s}{s-1} + s\int_1^{\infty}\frac{y\ (x) - x}{x^{s+1}}$$

i.e., $$-\frac{V'(s)}{V(s)} - \frac{s}{s-1} = s\int_1^{\infty}\frac{y\ (x) - x}{x^{s+1}} \qquad (14).$$

If ψ(x) = x + O($x^{1/2+\varepsilon}$) then the integral on the right side converges absolutely for Re(s) = σ > ½ and uniformly in any region of the form Re(s) = σ ≥ s_0 > ½, and hence, by a theorem of Weierstrass, represents a holomorphic (analytic) function of s in the region Re(s) = σ > ½ giving analytic continuation of the function on the left and proving that ζ(s) cannot vanish in this region.

We end this section by listing a few more statements which are equivalent to RH.

Theorem 3.2.3 The following are equivalent to the Riemann Hypothesis:

(i) θ(x) = x + O ($\sqrt{x}$ $\log^2$x)

(ii) π (x) =Li(x) + O ($\sqrt{x}$ logx)

(iii) M(x):= $\sum_{n\le x} m(n)$ = O($x^{1/2+\varepsilon}$) for any ε > 0.

Mertens conjectured on the basis numerical evidence that | M(x)| < $\sqrt{x}$ for all x. This conjecture, however, was refuted by Odlyzko and te Riele who proved that $\limsup \frac{M(x)}{\sqrt{x}} \ge 1.06$, and $\liminf \frac{M(x)}{\sqrt{x}} \le -1.009$. One more conjecture stronger than RH was refuted : Turan had proved that absence of zeros of partial sums of Dirichlet series of ζ(s) in the half plane Re(s) > 1 implies RH. However, in 1985, Voronin proved there exist x_0 and c >0 such that for x > x_0 the partial sum $\sum_{n\le x}\frac{1}{n^s}$ has a zero ρ with Reρ > 1 + (c / log x), refuting this conjecture.

3.3 Density Theorems and Gaps between Primes

The Riemann Hypothesis implies that for any x and some constant c > 0, the interval (x, x+h) with h ≥ c $\sqrt{x}\log^2 x$ will contain a prime. In the results about “large gaps” between primes and some other places, generally we can obtain weaker results independent of RH by applying a “Density Theorem”. A “Density Theorem” is a non-trivial bound on N(σ,T), the number of the zeros of the zeta-function ζ(s) in the region Re(s) ≥ σ, 0 ≤ |γ| ≤ T for some σ > 1/2. (Note that, RH implies $N(\sigma,T) = 0$ for $\sigma > ½$). We present two such theorems and one of their consequences (without proofs, which can be found, for example, in [KaVo]).

Theorem 3.3.1 For $½ \le \sigma \le 1$, $T \ge 2$, $N(\sigma,T) = O(T^{4\sigma(1-\sigma)} \log^{13}T)$.

Theorem 3.3.2(Huxley) For $½ \le \sigma \le 1$, $T \ge 2$, $N(\sigma,T) = O(T^{2.4(1-\sigma)} \log^{c}T)$ for some absolute constant $c > 0$.

Theorem 3.3.3 Suppose that for some $a \ge 2$, absolute constants c, $c_1 > 0$, $T \ge 2$, and all σ with $½ \le \sigma \le 1$, the density theorem $N(\sigma,T) \le c_1 T^{a(1-\sigma)} \log^{c}T$ holds. Then for $x \ge x_0 > 0$, $x \ge h \ge x^{1-(1/a)}\exp(\log^{0.8}x)$, the interval $(x, x+h)$ contains a prime.
In particular, by theorem 3.3.2 above, it follows that,

$$p_{n+1} < p_n + p_n^{7/12+\varepsilon} \qquad \text{for } n \ge n_0(\varepsilon) \qquad (15).$$

Here p_n denotes nth prime when primes are arranged in increasing order.

Chapter 4
Approximate Functional Equations and Computations of Zeros of z (s)

4.1 Approximate Functional Equations

In Sec. 1.4, we derived the following expression for the zeta-function ζ(s) for Re(s) > 0 (cf. (1.25)):

$$V(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{N^{1-s}}{s-1} - \frac{1}{2} N^{-s} + s\int_N^{\infty} \frac{r_1(u)du}{u^{s+1}} \tag{1.25}$$

where $\rho_1(x) := x - [x] - ½ = \{x\} - ½$.

In 1922, hardy and Littlewood applied techniques of replacing a trigonometric sums by a shorter sum and theory of trigonometric sums to obtain the following approximate equation for ζ(s) in the region Re(s) > 0.

Theorem 4.1.1 (Approximate Functional equation for z (s)) Suppose that $t \geq 2\pi$, and x, y are real numbers satisfying $x \geq h > 0$, $y \geq > 0$, $2\pi xy = t$. Then for $0 < \sigma \leq \sigma_0 \leq 2$, we have,

$$V(s) = V(s+it) = \sum_{n \leq y} \frac{1}{n^s} + \exp(-2iq(t)) \left(\frac{t}{2p} \right)^{1/2-s} \sum_{n \leq x} \frac{1}{n^{1-s}} + O(t^{1/2-s} x^{-1+s}) + O(y^{-s} \log t)$$

(1).

where $q(t) = t \log \sqrt{\frac{t}{2p}} - \frac{t}{2} - \frac{p}{8}$ (2).

and the constants in O-terms depend only on h and σ_0.

Definition The Hardy-Function Z(t) is defined by

$$Z(t) := e^{iq_1(t)} V\left(\frac{1}{2} + it \right) \tag{3}$$

where $e^{iq_1(t)} = p^{-it/2} \Gamma\left(\frac{1}{4} + \frac{it}{2} \right) \left| \Gamma\left(\frac{1}{4} + \frac{it}{2} \right) \right|^{-1}$ (4).

We have, then, $V\left(\frac{1}{2}+it\right)$ = $e^{-iq_1(t)}Z(t)$, |ζ(s)| = |Z(t)|.

Note that Z(t) is real for real t and the real zeros of Z(t) are the zeros of the zeta-function ζ(s) on the critical line. Also using Stirling's formula (Theorem 1.3.5), we see that for t ≥ 2,

$$q_1(t) = t\log\sqrt{\frac{t}{2p}} - \frac{t}{2} - \frac{p}{8} + \Delta(t) \qquad (5).$$

where $\Delta(t) = \frac{t}{4}\log\left(1+\frac{1}{4t^2}\right) + \frac{1}{4}\arctan\frac{1}{2t} - \frac{t}{2}\int_0^\infty \frac{r(u)}{\left(u+\frac{1}{4}\right)^2 + \frac{t^2}{4}}$ (6).

in which ρ(u) =1/2 - {u}. Δ (t) = O(1/t).

In fact, an application of Stirling's series gives,

$$q_1(t) = t\log\sqrt{\frac{t}{2p}} - \frac{t}{2} - \frac{p}{8} + \frac{1}{48t} - \frac{7}{5760t^3} + \ldots \qquad (7).$$

which, for large t, has a small error even if truncated at 3rd or 4th term.

The following approximate formula for the Hardy-Function Z(t), found by Siegel in1932 through study of an unpublished manuscript of Riemann is the basis of most of computations of values of the zeta-function ζ(s) on the critical line, esp. those relating to the zeros of ζ(s) on the critical line although it has been modified to calculate values of ζ(s) for s not on the critical line as well. Prior to rediscovery of Riemann's formula, the most widely used method for calculating values of ζ(s) was Euler-Maclaurin summation formula, which is a general method applicable to any function satisfying some regularity conditions and required order of |t| steps to calculate ζ(1/2+it). Riemann-Siegel formula is a very specialized method to evaluate ζ(s) on the critical line and requires only order $\sqrt{|t|}$ steps per evaluation.

Theorem 4.1.2. (Riemann-Siegel Formula) For t ≥ 2π,

$$Z(t) = 2\sum_{n\le\sqrt{t/2p}} \frac{1}{\sqrt{n}}\cos\left(q(t) - t\log n\right) + O(t^{-1/4}\log t) \qquad (8).$$

where $q(t) = t\log\sqrt{\frac{t}{2p}} - \frac{t}{2} - \frac{p}{8}$.

4.2 Zeros of z (s) on the Critical Line

Before discussing the calculations of the zeros of the zeta-function we list the proved theorems about the zeros of the zeta-function ζ(s) on the critical line in chronological order. (Recall: RH says that they are all the complex zeros of ζ(s).)

Theorem 4.2.1 (Hardy,1914) There are infinitely many zeros of the zeta-function $\zeta(s)$ on the critical line.

Theorem 4.2.2 (Hardy and Littlewood,1918) For $T \geq T_0(\varepsilon)$, $H \geq T^{1/4+\varepsilon}$, the interval (T, T+H) contains at least one odd order zero of Z(t).

Theorem 4.2.3 (Hardy and Littlewood,1921) For $T \geq T_0(\varepsilon)$, $H \geq T^{1/2+\varepsilon}$, the interval (T,T+H) contains at least c H odd order zeros of Z(t) for some $c = c(\varepsilon) > 0$.

Theorem 4.2.3 (Selberg,1942) For $T \geq T_0(\varepsilon)$, $H \geq T^{1/2+\varepsilon}$, the interval (T, T+H) contains at least c H log T odd order zeros of Z(t) for some $c = c(\varepsilon) > 0$.
(Note that this implies that at least a positive fraction of the complex zeros of the zeta-function $\zeta(s)$ are actually on the critical line.)

Theorem 4.2.4 (Levinson,1974) At least one third of the complex zeros of the zeta-function $\zeta(s)$ are on the critical line.

Theorem 4.2.5 (Moser,1976) For $T \geq T_0$, $H \geq T^{1/6}\log^2 T$, the interval (T, T+H) contains at least one odd order zeros of Z(t).

Theorem 4.2.6 (Moser,1980) For $T \geq T_0$, $H \geq T^{5/12}\log^2 T$, the interval (T, T+H) contains at least c H odd order zeros of Z(t) for some $c = c(\varepsilon) > 0$.

Theorem 4.2.7 (Karatsuba,1983) For $T \geq T_0$, $H \geq T^{5/32}\log^2 T$, the interval (T, T+H) contains at least one odd order zeros of Z(t).

Theorem 4.2.8 (Karatsuba,1985) Let $0 < \varepsilon \leq 0.001$. For $T \geq T_0(\varepsilon)$, $H = T^{127/82+\varepsilon}$, the interval (T, T+H) contains at least c H log T odd order zeros of Z(t) for some $c = c(\varepsilon) > 0$ (This statement was a conjecture of Selberg).

Theorem 4.2.9 (Conrey and Ghosh,1989) At least two-fifths of the complex zeros of the zeta-function $\zeta(s)$ are on the critical line.

We also mention that there are proved results which assert that "almost all" the complex zeros of the zeta-function $\zeta(s)$ are arbitrarily close to the critical line. To make precise the meaning of almost all we state two such results.

Theorem 4.2.10 Let $f(t) \to \infty$ as $x \to \infty$ then for any $\varepsilon > 0$ and $T \geq T_0(\varepsilon,f)$ at least $(1-\varepsilon)$ fraction of complex zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ in the region $0 \leq |\gamma| \leq T$ satisfy $|\beta-1/2| < f(t) / \log t$.

Theorem 4.2.11 The number J(T) of the complex zeros of $\zeta(s)$ in the region $0 \leq |\gamma| \leq T$ that do not satisfy $|\sigma-1/2| < B \log\log T / \log T$ is $O(T \log^{-(B/4)+1} T)$.

For proofs of these results, see [TiHB] or [KaVo].

4.3 Computations of Zeros of ζ (s)

Gram's "Law"

It is possible to verify by means of calculations that all the complex zeros of $\zeta(s)$ up to a certain "height" (since they are placed symmetrically about the real-axis, we don't need to go in "depth") in the critical strip are located exactly, not merely approximately, on the critical line(and are simple). Since the Hardy-function

$$Z(t) := e^{iq_1(t)} V(\frac{1}{2}+it)$$

is real for real t and its zeros correspond to the zeros of $\zeta(s)$ on the critical line, if $Z(t_1)$ and $Z(t_2)$ have opposite sign then Z(t) vanishes between t_1 and t_2 at least once and $\zeta(s)$ has a zero on the critical line between $½+it_1$ and $½ +it_2$. (Once we know that there is a zero in between we can find one with arbitrary precision. However, for the purpose of verification of the Riemann Hypothesis, it is not the exact location of the zero but rather its existence that is important.) It must be noted that there can be more than one zero between t_1 and t_2, and we do need a method to ensure that there are no more zeros than we have found. One such method that we shall discuss was devised by Turing. First we will discuss an empirical "law" based on observations made by Gram (1903) who calculated first 15 zeros of $\zeta(s)$ in the critical strip. Gram made the following observations:

1. Re($\zeta(1/2+it)$) has a tendency to be positive (for t positive) while the values of Im($\zeta(1/2+it)$) are distributed more evenly between positive and negative ones.
2. The zeros of Z(t) tend to alternate with the zeros of sin $(q_1(t))$.

Although these "laws" are based on purely empirical results at low values of t, they are the foundations on which modern root counting strategies are based. For $n \geq -1$, we define g_n to be the real number satisfying

$$q_1(g_n) = n \qquad (9).$$

and call it the nth Gram point. Gram's second empirical observation is known as Gram's "Law" and is, then, stated in the form

$$(-1)^n Z(g_n) > 0 \qquad (10^*).$$

This "law" does fail infinitely often but reduces the verification of the Riemann Hypothesis to the more infrequent cases where it fails to hold. Call a Gram point g_n "good" if it satisfies Gram's law, and "bad" otherwise, and define a Gram block to be an interval $[g_n, g_{n+k})$, $k \geq 1$ such that g_n and g_{n+k} are "good" Gram points but all that are strictly inside the interval are "bad". We then count the "good" Gram points and the number of zeros of Z(t) in each Gram block. Note once again that although there is at least one root of Z(t) between two good (or two bad) Gram points but there can be more than one, and such cases, though rare, have been encountered. This is where Turing's method comes into picture. It gives an upper bound on the number of roots in our range, and we compare the number of roots found with it to make sure that indeed we have found all the roots in the range.

Turing's Method

Once we have calculated roots of Z(t) on the critical line up to some height T, we must be able to determine whether these are all the roots in the critical strip up to this height. We need a precise formula for N(T) to check if, basing our strategy on Gram's "Law", we have indeed found all the zeros.

Theorem 4.3.1 $$N(T) = \frac{1}{p} q_1(t) + 1 + \frac{1}{p}\Delta(\arg V(s)) = F(T) + S(T) + O\left(\frac{1}{T}\right) \qquad (11).$$

where $$F(T) := \frac{T}{2p}\log\frac{T}{2p} - \frac{T}{2p} + \frac{7}{8} \qquad (12).$$

and $$S(T) := \frac{1}{p}\Delta(\arg V(s)) = \frac{1}{p}\operatorname{Im}\left(\int_C \frac{\dot{V}(s)}{V(s)}\,ds\right) \qquad (13).$$

is continuous variation of arg(ζ(s)) along the path C which consists of straight line segments joining 2, 2+iT, and ½+iT. We assume here that no zero of ζ(s) lies on the line t = T.

For a proof see [Ed] or [TiHB]

If Re(ζ(s)) is non-zero (and hence > 0 as ζ(2+it) > 0) along the path C then the integrand in (13) has an anti-derivative which may be bounded to give

$$S(T) = \left|\frac{1}{p}\operatorname{Im}\left(\int_C \frac{\dot{V}(s)}{V(s)}\,ds\right)\right| < \frac{1}{2} \qquad (14).$$

and N(T) will be the integer nearest to $\frac{1}{p}q_1(t) + 1$. This gives exact value of N(T) if only we could prove that Re(ζ(s)) > 0 on the line segment joining 2+iT and ½+iT. The main problem is to bound S(T) = N(T) - $\frac{1}{p}q_1(t) - 1$. Riemann-von Mangoldt formula for N(T) (Theorem 2.3.1) basically says that S(T) = O(log T). Littlewood proved that $\int_0^T S(T)$ = O(log T) from which Turing was able to get the bound

$$\left|\int_{T_1}^{T_2} S(t)\,dt\right| \le 2.30 + 0.128\log\left(\frac{T_2}{2p}\right) \qquad \text{for } 168\pi < T_1 < T_2 \qquad (15).$$

Turing then used this expression to bound $S(g_n)$ for certain Gram points g_n, thereby proving that $N(g_n) \le n+1$. If we have found n+1 zeros of Z(t) with $t \le g_n$, this will not only verify the Riemann Hypothesis up to height $t = g_n$ in the critical strip but also prove that all the zeros up to this height are, in fact, simple zeros. (In fact, Turing's method can be used to bound $S(g_n)$ from below also for certain Gram points g_n, and will give us exact value of $N(g_n)$ for such g_n.) Furthermore, Turing's method doesn't require any extra computation other than what would be necessary anyway to detect a change in the sign of Z(t). Let $[g_n, g_{n+k})$, $k \geq 1$, be a Gram Block and suppose that h_j, j = 1, 2,…, k - 1 (we may take $h_0 = 0$ and $h_k = 0$) are numbers such that $g_{n+j}+ h_j$ increase with j and $(-1)^{n+j} Z(g_{n+j}+ h_j) > 0$ for all j. We try to find h_j as small as possible, and Turing proved that if h_j are not too large then $S(g_n) =0$, and hence, $N(g_n) = n +1$. First note that $S(g_n)$ must be an even integer because firstly $S(g_n) = N(g_n) - \theta_1(g_n) -1 = N(g_n) - n +1$ is an integer and the parity of the number of roots on the critical line up to height g_n (counted with multiplicity) is given by sign of $-Z(g_n)$ which is $(-1)^{n+1}$ ($g_0 = 17.8456$, $Z(g_0) > 0$, and there is exactly one zero of Z(t) below this height viz. t = 14.13) while the zeros off the critical line, if any(!), must occur in pairs. Turing proved that

$$S(g_n) \leq 1 + \frac{2.30 + 0.128\log(g_{n+k}/2p) + \sum_{j=1}^{k-1} h_j}{g_{n+k} - g_n} \qquad \text{(16a).}$$

(Similarly, in the other direction, if $[g_{n-k},g_n)$ is a Gram Block then

$$S(g_n) \geq -1 - \frac{2.30 + 0.128\log(g_{n+k}/2p) - \sum_{j=1}^{k-1} h_j}{g_{n+k} - g_n} \qquad \text{(16b).)}$$

If the number h_j are small then $S(g_n)$ is bounded by 2 and hence must be zero. In all actual computations done h_j have been found so as to satisfy the inequalities above. Odlyzko has developed algorithms which verify the Riemann Hypothesis in a large region around 10^{22}th and 10^{23}rd zeros of $\zeta(s)$. He used Turing's method to pinpoint the 10^{22}th and 10^{23}rd zeros.

History of Computation of the Zeros of ζ (s)

In 1859, Riemann published his 8-page paper, his only published work on Number Theory, which contained many statements that were not supported by rigorous proofs and took many years to be proved later. In 1932, when Siegel published an account the work on the zeta-function found in Riemann's private papers, it turned out to be an important event in the history of the theory of his zeta-function, and showed clearly that behind Riemann's speculations was an extensive analysis and calculations. Riemann seems to have calculated at least first six zeros of the zeta-function and his manuscript essentially contains the asymptotic formula, known as the Riemann-Siegel formula, to calculate Z(t), which is the basis of all modern computations of Z(t) and then the zeros of ζ(s).

In 1903, Gram calculated first 15 zeros of the zeta-function and formulated his empirical law called Gram's "law". (Z(t) usually changes sign in each Gram interval $G_n = [g_n, g_{n+1})$.) This "law," fails infinitely often, although the first failure does not occur until the Gram point $g_{125} = 280.80$. In 1968, Rosser, Yohe and Schoenfeld introduced the concept of Gram blocks and Rosser's "rule" (Z(t) has at least k zeros in Gram block B_n.)to handle the failures of Gram's "law". This rule also fails infinitely often (R. S. Lehman 1970),the first exception being $B_{13,999,525}$, $g_{13,999,525} = 6,820,050.98$, but it is still an extremely useful heuristic.

At the end, we present here a table of some big contributions in calculation of the complex zeros of the zeta-function. It should be mentioned here that, in 2001, Sebastian Wedeniwski of IBM founded ZetaGrid which uses the Fortran-Code of J. van de Lune, H. J. J. te Riele, D. T. Winter and since sept. 2002 it has been made available on the internet for free download at http://www.zetagrid.net.

Year	Author	Number of Zeros r calculated with Im(r) > 0.
1903	J. P. Gram	15
1914	R. J. Backlund	79
1925	J. I. Hutchinson	138
1935	E. C. Titchmarsh	1,041
1953	A. M. Turing	1,104
1955	D. H. Lehmer	10,000
1956	D. H. Lehmer	25,000
1958	N. A. Meller	35,337
1966	R. S. Lehman	250,000
1968	J. B. Rosser, J. M. Yohe, L. Schoenfeld	3,500,000
1977	R. P. Brent	40,000,000
1979	R. P. Brent	81,000,001
1982	R. P. Brent, J. van de Lune, H. J. J. te Riele, D. T. Winter	200,000,001
1983	J. van de Lune, H. J. J. te Riele	300,000,001
1986	J. van de Lune, H. J. J. te Riele, D. T. Winter	1,500,000,001
1989	A. M. Odlyzko	70,000,000 (starting from 10^{20})
1992	A. M. Odlyzko	175,000,000 (starting from 10^{20})
2001	A. M. Odlyzko	10,000,000,000 (starting from 10^{22})
2001	J. van de Lune	10,000,000,000
2002	A. M. Odlyzko	20,000,000,000 (starting from 10^{23})
2002	S. Wedeniwski (ZetaGrid)	75,000,000,000
2003	S. Wedeniwski (ZetaGrid)	570,000,000,000 (on Nov. 20th)

I don’t believe in God, but I believe in Riemann Hypothesis. For the latter there are more than 400,000,000,000 reasons (to believe).

- Manoj Verma.

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